On decidability of the decomposability problem for finite theories*

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Abstract

We consider the decomposability problem for elementary theories, i.e. the problem of deciding whether a theory has a nontrivial representation as a union of two (or several) theories in disjoint signatures. For finite universal Horn theories, we prove that the decomposability problem is Σ_1^0–complete and, thus, undecidable. We also demonstrate that the decomposability problem is decidable for finite theories in signatures consisting only of monadic predicates and constants.

The interest in studying partitioning of theories is connected with the applications of component methods in automated theorem proving [1] and the development of terminological systems [2, 3, 4, 5]. The study of the decomposability property for elementary theories was initiated in [6], where a decomposability criterion was formulated and it was proven that each elementary theory has a unique decomposition into indecomposable components. These results were then extended to a more general Δ–decomposability property of theories in a broad class of logical calculi [7].

The natural question is whether the decomposability property can be decided effectively. It turns out that for a number of finite signatures, the decomposability problem is Σ_1^0–complete and, thus, undecidable. This is true already for finite universal Horn theories. However, if the signature consists only of monadic predicates and constants then the decomposability problem is decidable. These results are proven in this paper. All basic definitions of the article are rather standard and can be found, for instance, in [8, 9, 10].

Let us start with some notation. The symbol ω stands commonly for the set of all natural numbers and $\{0,1\}^n$ denotes the set of all n—tuples consisting of zeros and ones. Given a signature Σ, we denote the first-order language of Σ by L_Σ. If $\mathcal{T}$ is a set of first-order formulas then $\mathtt{sig}(\mathcal{T})$ stands for the signature of $\mathcal{T}$. By a theory in signature Σ, we mean any arbitrary set of formulas in signature Σ.

Let us recall the notion of a decomposable theory.

*This work was carried out within the COMO project of DFG (GZ: 436 RUS 113/829/0-1) and Russian Foundation for Basic Research (project No. 05-01-04003-NNIO_a).

Definition 1 ([6]) *Let $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{T}$ be theories satisfying the following:*

1. $\mathtt{sig}(\mathcal{S}_0) \neq \varnothing \neq \mathtt{sig}(\mathcal{S}_1)$;
2. $\mathtt{sig}(\mathcal{S}_0) \cap \mathtt{sig}(\mathcal{S}_1) = \varnothing$, $\mathtt{sig}(\mathcal{S}_0) \cup \mathtt{sig}(\mathcal{S}_1) = \mathtt{sig}(\mathcal{T})$;
3. *$\mathcal{T}$ is equivalent to $\mathcal{S}_0 \cup \mathcal{S}_1$.*

Then we say that $\mathcal{T}$ is decomposable into theories $\mathcal{S}_0$ and $\mathcal{S}_1$. The pair $\langle \mathcal{S}_0, \mathcal{S}_1 \rangle$ is called decomposition of $\mathcal{T}$; we use the notation $T = \mathcal{S}_0 \uplus \mathcal{S}_1$. The theories $\mathcal{S}_0$ and $\mathcal{S}_1$ are called decomposition components and the signatures $\mathtt{sig}(\mathcal{S}_0)$ and $\mathtt{sig}(\mathcal{S}_1)$ are called signature decomposition components of $\mathcal{T}$. A theory is called decomposable if it has at least one decomposition. A sentence φ is called decomposable if it axiomatizes a decomposable theory.

Let us note the following straightforward model–theoretic decomposability criterion:

Proposition 1 *Let a signature Σ be the union of non-empty disjoint signatures Σ_0 and Σ_1. For $i = 0, 1$, let $\mathcal{T}_i$ be the set of all formulas in signature Σ_i entailed by $\mathcal{T}$. Then the following are equivalent:*

1. *$\mathcal{T}$ is decomposable into components in Σ_0 and Σ_1;*
2. *for each model $\mathfrak{M}$ of signature Σ, we have $\mathfrak{M} \models \mathcal{T}$ iff for $i = 0, 1$, the restriction of $\mathfrak{M}$ to Σ_i is a model of $\mathcal{T}_i$.*

We consider only *effective signatures* in which to every symbol s there is assigned the unique natural number $i \in \omega$ such that the set of all these numbers is computably enumerable and the arity of s can be effectively computed from i. If Σ is an effective signature, then it is possible to introduce a Gödel numbering γ of all formulas in signature Σ such that the mappings $\gamma : \{\varphi \mid \varphi \text{ is a formula in } \Sigma\} \to \omega$ and γ^{-1} are both computable. Having γ, one can define a numbering of all finite sets of formulas. Let D be a numbering of all finite subsets of ω defined as follows:

$$D_n = \{a_0 < a_1 < \ldots < a_{k-1}\} \quad \Leftrightarrow \quad n = \sum_{i<k} 2^{a_i}.$$

Then to a finite set of formulas $\{\varphi_{i_0}, \ldots \varphi_{i_j}\}$ we assign some number n, if $D_n = \{i_0, \ldots i_j\}$. If n is the number for a finite set $\mathcal{T}$ of formulas then we call n the *index* of $\mathcal{T}$.

For a partial computable function f, by $\mathtt{range}\,(f)$ we denote the range of f.

Definition 2 The decomposability problem for finite sets of sentences in signature Σ *is the set of indices of finite decomposable theories in signature Σ.*

In other words, this is the problem to decide whether a given finite set of sentences in signature Σ is decomposable.

Finite universal Horn theories

A theory axiomatized by a finite set of quasi-identities is called *finite universal Horn theory.* It is known that these axiomatizations correspond to the class of logic programs [11]. For the theories of this kind, we will write axioms without quantifiers assuming the universal quantification of all variables and use the left-sided style of writing implications which is common to logic programming.

Theorem 1 *There exists a finite signature Σ such that the decomposability problem for finite universal Horn theories in Σ is Σ^0_1–complete and, thus, undecidable.*

Proof. We demonstrate that there exists an algorithm which, given $m \in \omega$ and the Gödel number of a primitive computable function f, computes the index of a finite universal Horn theory $\mathcal{S}_m$ such that $\mathcal{S}_m$ is decomposable iff $m \in \mathtt{range}\,(f)$.

Let us fix a finite sequence $f_1, \ldots, f_k$ of functions, where $f_k \equiv f$ and for each $i \in \{1, \ldots, k\}$ one of the following cases holds:

1. f_i is either a projection function $I^n_m(x_1, \ldots, x_n) = x_m$, $1 \leqslant m \leqslant n$, the zero function $0(x) = 0$, or the successor function $s(x) = x + 1$;
2. f_i is a superposition of some elements in this sequence with numbers smaller than i;
3. f_i is obtained by primitive recursion from some elements in this sequence with numbers smaller than i.

The sequence $f_1, \ldots, f_k$ can be viewed as a computation schema for f. For each $i \in \{1, \ldots, k\}$ we fix one of the cases above for f_i.

We now prove a modified version of the result on the representability of partial computable functions by logic programs (Theorem 9.6 in [11]). In particular, the extra argument $\mathbf{s}(\mathbf{0})$ appears in the proof to make the representing theory indecomposable. We need a proof only for primitive computable functions.

Consider the finite universal Horn theory $\mathcal{T}$ defined as follows: the language of $\mathcal{T}$ contains a unary operation symbol $\mathbf{s}$, a constant symbol $\mathbf{0}$, and for each $i \in \{1, \ldots, k\}$ a $n+2$–ary predicate symbol P_i, where n is the number of arguments of f_i. For each $i \in \{1, \ldots, k\}$, the theory $\mathcal{T}$ contains some axiom φ_i of the following form:

Case 1: *f_i is $I^n_m(x_1, \ldots, x_n)$.*
Then φ_i is the universal closure of $P_i(x_1, \ldots, x_n, x_m, \mathbf{s}(\mathbf{0}))$.

Case 2: *f_i is $0(x)$.*
Then φ_i is the universal closure of $P_i(x, \mathbf{0}, \mathbf{s}(\mathbf{0}))$.

Case 3: *f_i is $s(x)$.*
Then φ_i is the universal closure of $P_i(x, \mathbf{s}(x), \mathbf{s}(\mathbf{0}))$.

Case 4: *$f_i(\bar{x}) = f_j(f_{l_1}(\bar{x}), \ldots, f_{l_q}(\bar{x}))$, where $j, l_1, \ldots, l_q < i$.*

In this case φ_i is the universal closure of the formula

$$P_i(\bar{x}, y, \mathbf{s}(\mathbf{0})) \leftarrow \bigwedge_{p=1}^{q} P_{l_p}(\bar{x}, z_p, \mathbf{s}(\mathbf{0})) \wedge P_j(z_1, \ldots, z_q, y, \mathbf{s}(\mathbf{0}))$$

Case 5: *$f_i(\bar{x}, y)$ is obtained by primitive recursion from $f_j(\bar{x})$ and $f_l(\bar{x}, y, z)$, $j, l < i$; namely, it is defined by the equalities $f_i(\bar{x}, 0) = f_j(\bar{x})$, $f_i(\bar{x}, y+1) = f_l(\bar{x}, y, f_i(\bar{x}, y))$.*

In this case φ_i is the conjunction of the universal closure of

$$P_i(\bar{x}, \mathbf{0}, t, \mathbf{s}(\mathbf{0})) \leftarrow P_j(\bar{x}, t, \mathbf{s}(\mathbf{0}))$$

and the universal closure of

$$P_i(\bar{x}, \mathbf{s}(y), t, \mathbf{s}(\mathbf{0})) \leftarrow P_l(\bar{x}, y, u, t, \mathbf{s}(\mathbf{0})) \wedge P_i(\bar{x}, y, u, \mathbf{s}(\mathbf{0})).$$

The definition of $\mathcal{T}$ is complete.

Lemma 1 *For all natural numbers $x_1, \ldots, x_n, y$, we have*

$$\mathcal{T} \vdash P_k\left(\mathbf{s}^{x_1}(\mathbf{0}), \ldots, \mathbf{s}^{x_n}(\mathbf{0}), \mathbf{s}^{y}(\mathbf{0}), \mathbf{s}(\mathbf{0})\right) \Leftrightarrow f(x_1, \ldots, x_n) = y.$$

Proof of the lemma.
($\Leftarrow$) follows easily from the definition of $\mathcal{T}$.

($\Rightarrow$) We define a model[1] $\mathfrak{M}$, which will play an important role in the rest of the proof of Theorem 1. Let the universe of $\mathfrak{M}$ be the set of all natural numbers, let 0 be the zero constant of $\mathfrak{M}$, and let s be the operation of $\mathfrak{M}$ defined in the natural way: $s(m) = m + 1$. Moreover, if f_i is m–ary then we put

$$\mathfrak{M} \models P_i(x_1, \ldots, x_m, y, z) \Leftrightarrow (f_i(x_1, \ldots, x_m) = y) \wedge (z = 1).$$

Clearly, $\mathfrak{M} \models \mathcal{T}$.

Suppose that

$$\mathcal{T} \vdash P_k(\mathbf{s}^{x_1}(\mathbf{0}), \ldots, \mathbf{s}^{x_n}(\mathbf{0}), \mathbf{s}^{y}(\mathbf{0}), \mathbf{s}(\mathbf{0})).$$

Then $\mathfrak{M} \models \mathcal{T}$ implies

$$\mathfrak{M} \models P_k(\mathbf{s}^{x_1}(\mathbf{0}), \ldots, \mathbf{s}^{x_n}(\mathbf{0}), \mathbf{s}^{y}(\mathbf{0}), \mathbf{s}(\mathbf{0})),$$

i.e., $\mathfrak{M} \models P_k(x_1, \ldots, x_n, y, 1)$, which yields $f(x_1, \ldots, x_n) = y$ by the definition of $\mathfrak{M}$. $\Box$

In what follows, we require the auxiliary definition and lemma that are formulated below.

[1] In terms of universal algebra, $\mathfrak{M}$ will be the *free algebra of rank* 0 *in the quasivariety defined by* $\mathcal{T}$.

Let M be a set and π be an arbitrary permutation of M. Let $Q \subseteq M^k$ be a predicate on M. Then $Q^\pi = \{\langle\pi(a_1), \dots, \pi(a_k)\rangle \mid \langle a_1, \dots, a_k\rangle \in Q\}$ is called the *predicate conjugate with* Q by π. If $F : M^k \to M$ is an operation on M, then the *operation conjugate with* F by π is defined as $F^\pi(\pi(a_1), \dots, \pi(a_k)) = \pi(b) \Leftrightarrow F(a_1, \dots, a_k) = b$. The *element conjugate with* $a \in M$ by π is $a^\pi = \pi(a)$.

The following property of decomposable theories is rather straightforward:

Lemma 2 *Let $\mathcal{T} = \mathcal{S}_1 \uplus \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are theories in signatures Σ_1 and Σ_2, respectively. Let $\mathfrak{M} \models \mathcal{T}$ and let π be an arbitrary permutation of the universe of $\mathfrak{M}$. Consider the model $\mathfrak{M}^\pi$ obtained from $\mathfrak{M}$ by replacing all operations, predicates and constants corresponding to Σ_1 with those conjugate by π. Then $\mathfrak{M}^\pi \models \mathcal{T}$.*

We proceed with the proof of Theorem 1. The important property of the theory $\mathcal{T}$ is that it is indecomposable under some natural extensions:

Lemma 3 *Suppose $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{T}'$ is satisfied in some expansion $\bar{\mathfrak{M}}$ of the model $\mathfrak{M}$. Assume $\mathcal{T}' = \mathcal{T}_0 \uplus \mathcal{T}_1$. Then either $\mathtt{sig}\,(\mathcal{T}) \subseteq \mathtt{sig}\,(\mathcal{T}_0)$ or $\mathtt{sig}\,(\mathcal{T}) \subseteq \mathtt{sig}\,(\mathcal{T}_1)$.*

Proof of the lemma. We prove by induction on $i \in \{1, \dots, k\}$ that the symbols P_i, $\mathbf{s}$, and $\mathbf{0}$ must be in the same signature component of the decomposition.

Assume this statement is true for all $l < i$, for some $i \leqslant k$. Prove it to be true for i. Consider the several cases:

Case 1: *f_i is a function* $I^n_m(x_1, \dots, x_n) = x_m$.

Subcase 1.1: $\mathbf{s}, \mathbf{0} \in \mathtt{sig}\,(\mathcal{T}_u)$ *and* $P_i \in \mathtt{sig}\,(\mathcal{T}_{1-u})$, $u \in \{0, 1\}$.

Take a permutation π of the universe of $\bar{\mathfrak{M}}$ such that $s^\pi(0^\pi) \neq 1$. Let $\bar{\mathfrak{M}}^\pi$ be the model obtained from $\bar{\mathfrak{M}}$ by replacing with the conjugate ones all operations, predicates and constants corresponding to the symbols of $\mathtt{sig}\,(\mathcal{T}_u)$. We have $\bar{\mathfrak{M}} \models \mathcal{T}'$, but clearly, $\bar{\mathfrak{M}}^\pi \not\models \forall \bar{x}\, P_i(\bar{x}, x_m, \mathbf{s}(\mathbf{0}))$, while the latter sentence belongs to $\mathcal{T} \subseteq \mathcal{T}'$. This contradicts Lemma 2, and so this subcase is impossible.

Subcase 1.2: $\mathbf{s}, P_i \in \mathtt{sig}\,(\mathcal{T}_u)$ *and* $\mathbf{0} \in \mathtt{sig}\,(\mathcal{T}_{1-u})$, $u \in \{0, 1\}$.

Choose a permutation π of the universe of $\bar{\mathfrak{M}}$ such that $0^\pi \neq 0$. An argument similar to the one above shows that this subcase is impossible.

Subcase 1.3: $\mathbf{0}, P_i \in \mathtt{sig}\,(\mathcal{T}_u)$ *and* $\mathbf{s} \in \mathtt{sig}\,(\mathcal{T}_{1-u})$, $u \in \{0, 1\}$.

It suffices to take a permutation π of the universe of $\mathfrak{M}$ with the property $s^\pi(0) \neq 1$ to show that this subcase is impossible either.

It follows from the subcases considered that in Case 1, all symbols $\mathbf{0}$, $\mathbf{s}$, and P_i must belong to the same signature component of the decomposition.

Case 2: *f_i is the function* $0(x) = 0$.

Subcase 2.1: $\mathbf{0}$ *and* P_i *belong to different signature components.*

Take a permutation π of the universe of $\bar{\mathfrak{M}}$ such that $0^\pi \neq 0$. We have $\bar{\mathfrak{M}} \models \mathcal{T}'$, but clearly $\bar{\mathfrak{M}}^\pi \not\models \forall x P_i(x, \mathbf{0}, \mathbf{s}(\mathbf{0}))$, while the latter sentence belongs to $\mathcal{T}'$. This contradicts Lemma 2; hence, this subcase is impossible.

Subcase 2.2: $\mathbf{0}, P_i \in \mathtt{sig}\,(\mathcal{T}_u)$ *and* $\mathbf{s} \in \mathtt{sig}\,(\mathcal{T}_{1-u}),\ \ u \in \{0, 1\}$.

Choose a permutation π such that $s^\pi(0) \neq 1$ and use the same trick as above to demonstrate that this subcase is impossible either.

We conclude that in Case 2, all symbols $\mathbf{0}$, $\mathbf{s}$, and P_i must belong to the same signature component of the decomposition.

Case 3: *f_i is the function* $s(x) = x + 1$.

Subcase 3.1: $\mathbf{s}$ *and P_i fall into different signature components.*

Take a permutation π of the universe of $\bar{\mathfrak{M}}$ such that $s^\pi \neq s$ as operations. By Lemma 2, we have $\bar{\mathfrak{M}} \models \mathcal{T}'$, but clearly $\bar{\mathfrak{M}}^\pi \not\models \forall x\, P_i(x, \mathbf{s}(x), \mathbf{s}(\mathbf{0}))$, while the latter sentence belongs to $\mathcal{T}'$. This contradicts Lemma 2; thus, this subcase is impossible.

Subcase 3.2: $\mathbf{s}, P_i \in \mathtt{sig}\,(\mathcal{T}_u)$ *and* $\mathbf{0} \in \mathtt{sig}\,(\mathcal{T}_{1-u}),\ u \in \{0, 1\}$.

Take a permutation π of the universe of $\bar{\mathfrak{M}}$ such that $0^\pi \neq 0$ and use the same trick as above to show that this case is impossible either.

Case 4: *f_i is obtained by superposition or primitive recursion from some functions with numbers smaller than i.*

By the induction hypothesis, the symbols $\mathbf{s}$, $\mathbf{0}$, and P_j, $j < i$ are in the same signature component of the decomposition. Suppose that P_i belongs to another signature component, i.e., we have

$\mathbf{s}, \mathbf{0} \in \mathtt{sig}\,(\mathcal{T}_u)$ *and* $P_i \in \mathtt{sig}\,(\mathcal{T}_{1-u}),\ \ u \in \{0, 1\}$.

Take a permutation π of the universe of $\bar{\mathfrak{M}}$ such that $\pi(1) \neq 1$. Let $\bar{\mathfrak{M}}^\pi$ be the model obtained from $\bar{\mathfrak{M}}$ by replacing with the conjugate ones all operations, predicates and constants corresponding to the symbols of $\mathtt{sig}\,(\mathcal{T}_u)$. Take arbitrary $x_1, \ldots, x_n \in \omega$. By Lemma 1, for $y = f_i(x_1, \ldots, x_n)$ we have

$$\mathcal{T} \vdash P_i(\mathbf{s}^{x_1}(\mathbf{0}), \ldots, \mathbf{s}^{x_n}(\mathbf{0}), \mathbf{s}^y(\mathbf{0}), \mathbf{s}(\mathbf{0})),$$

but at the same time

$$\bar{\mathfrak{M}}^\pi \not\models P_i(\mathbf{s}^{x_1}(\mathbf{0}), \ldots, \mathbf{s}^{x_n}(\mathbf{0}), \mathbf{s}^y(\mathbf{0}), \mathbf{s}(\mathbf{0})).$$

This contradicts Lemma 2; hence, P_i must be in the same signature component together with the symbols $\mathbf{s}$, $\mathbf{0}$, and P_j for $j < i$. $\square$

Now define the theory $\mathcal{S}_m$ as:

$$\mathcal{S}_m = \mathcal{T} \cup \{\forall xy\, (Q(y) \leftarrow P_k(x, \mathbf{s}^m(\mathbf{0}), \mathbf{s}(\mathbf{0})))\},$$

where $\mathcal{T}$ is the theory constructed above, P_k is the predicate corresponding to the function f, and Q is a new unary predicate. Note that the index of $\mathcal{S}_m$ can be uniformly computed by m and the Gödel number for f.

Lemma 4 *The theory $\mathcal{S}_m$ is decomposable iff* $m \in \mathtt{range}\,(f)$.

Proof of the lemma.
($\Leftarrow$) By Lemma 1, $m \in \texttt{range}\,(f)$ yields $\mathcal{S}_m = \mathcal{T} \biguplus \{\forall y\, Q(y)\}$.

($\Rightarrow$) Assume that $m \notin \texttt{range}\,(f)$ and $\mathcal{S}_m$ is decomposable: $\mathcal{S}_m = \mathcal{T}_0 \uplus \mathcal{T}_1$. Let $\mathfrak{M}_0$ be the expansion of the model $\mathfrak{M}$ with the empty predicate Q, i.e., $\mathfrak{M}_0 \models \forall y\, \neg Q(y)$. Then, clearly, $\mathfrak{M}_0 \models \mathcal{S}_m$. By Lemma 3, we may assume without loss of generality that $\texttt{sig}\,(\mathcal{T}_0) = \{Q\}$. We have $\mathfrak{M}_0 \models \mathcal{T}_0$; thus, $\mathcal{T}_0$ is satisfied in any countable structure with the empty predicate Q.

Define $\mathfrak{M}_1$ as the model obtained from $\mathfrak{M}$ by extending the predicate P_k with the tuple $\langle 0, m, 1\rangle$ and adding the new predicate Q that is true on all elements. We have $\mathfrak{M}_1 \models \mathcal{S}_m$; hence, $\mathfrak{M}_1 \models \mathcal{T}_1$.

Consider the reduction $\mathfrak{M}^*$ of $\mathfrak{M}_1$ in which Q is replaced with the empty predicate. We have $\mathfrak{M}^* \models \mathcal{T}_0$ and $\mathfrak{M}^* \models \mathcal{T}_1$. Hence, from $\mathcal{S}_m = \mathcal{T}_0 \uplus \mathcal{T}_1$ we see that $\mathfrak{M}^* \models \mathcal{S}_m$. On the other hand, the conditions $\mathfrak{M}^* \models P_k(0, \mathbf{s}^m(\mathbf{0}), \mathbf{s}(\mathbf{0}))$ and $\mathfrak{M}^* \models \forall y\, \neg Q(y)$ imply

$$\mathfrak{M}^* \not\models \forall xy (Q(y) \leftarrow P_k(x, \mathbf{s}^m(\mathbf{0}), \mathbf{s}(\mathbf{0}))),$$

i.e., $\mathfrak{M}^* \not\models \mathcal{S}_m$; the contradiction yields that the theory $\mathcal{S}_m$ is indecomposable. $\square$

Take a primitive computable function f such that its range is a Σ^0_1–complete set (for a justification that f can be chosen as primitive computable, the reader is referred to [12], Section 4.2, Proposition 4.4). By applying the construction above, we obtain a family of theories $\mathcal{F} = \{\mathcal{S}_m \mid m \in \omega\}$ such that the range of f 1-reduces to the set of decomposable theories from $\mathcal{F}$. This completes the proof of Theorem 1 and yields Σ^0_1–completeness of the decomposability problem for finite universal Horn theories. $\square$

Theories in simple signatures

Let us call a signature *simple,* if it is finite and consists only of monadic predicate symbols and constants.

Theorem 2 *There exists an algorithm to decide for every simple signature Σ and every sentence $\varphi \in L_\Sigma$ whether the theory $\{\varphi\}$ is decomposable.*

The theorem will be a consequence of the following

Proposition 2
1. *There exists an effective procedure that, given simple signatures σ and τ with $\tau \subseteq \sigma$ and a satisfiable sentence $\varphi \in L_\sigma$, outputs a sentence $\varphi^\tau \in L_\tau$ such that*
   - *(a) $\varphi \vdash \varphi^\tau$;*
   - *(b) for each $\psi \in L_\tau$, we have $\varphi^\tau \vdash \psi$, if $\varphi \vdash \psi$.*
2. *The theory of the class of all models of any arbitrary simple signature is decidable. Moreover, the corresponding decision procedure can be effectively constructed from a given simple signature.*

Let us demonstrate that Theorem 2 indeed follows from Proposition 2. First, by using decidability, we check whether the sentence φ is unsatisfiable. If yes, then the theory $\{\varphi\}$ is, clearly, decomposable when the signature contains more than one symbol, and indecomposable otherwise. If φ is satisfiable, then it suffices to prove that the theory $\{\varphi\}$ is decomposable into components with signatures σ_0 and σ_1 iff

$$\{\varphi\} = \{\varphi^{\sigma_0}\} \uplus \{\varphi^{\sigma_1}\}. \tag{1}$$

Proof. Let $\{\varphi\}$ be decomposable into components with signatures σ_0 and σ_1. Then, by compactness, there exist sentences $\varphi_0 \in L_{\sigma_0}$ and $\varphi_1 \in L_{\sigma_1}$ such that $\{\varphi\} = \{\varphi_0\} \uplus \{\varphi_1\}$. We have

$$\varphi \vdash \varphi^{\sigma_0}, \quad \varphi \vdash \varphi^{\sigma_1}. \tag{2}$$

On the other hand,

$$\varphi^{\sigma_0} \vdash \varphi_0, \quad \varphi^{\sigma_1} \vdash \varphi_1. \tag{3}$$

As $\varphi_0 \wedge \varphi_1 \vdash \varphi$, the conditions (2) and (3) give the required (1). The converse is trivial. As the theory of all models of a simple signature is decidable, the condition (1) can be checked effectively and so the statement of Theorem 2 is proved. $\square$

We now prove Proposition 2 and describe first some normal forms of sentences in simple signatures. Let

$$\sigma = \left\langle (P_i)_{i<l}\,; (c_i)_{i<q} \right\rangle; \quad l, q < \omega$$

be a simple signature. Let A be an arbitrary subset of constants from σ (not necessarily all of them) and E be an arbitrary equivalence on A. We define a sentence η_E stating this equivalence as

$$\eta_E \quad = \bigwedge_{\langle c,k\rangle \in E} (c = k) \ \wedge \bigwedge_{c,k\in A, \langle c,k\rangle \notin E} (c \neq k).$$

Given a monadic predicate P, we define $P^0(x) = \neg P(x)$ and $P^1(x) = P(x)$. Given $\varepsilon \in 2^m$ and monadic predicates $P_0,\ldots,P_{m-1}$, we put $P^\varepsilon(x) = \bigwedge_{i<m} P_i^{\varepsilon_i}(x)$. For a set A of constants, the notation $x \notin A$ will abbreviate the formula $\bigwedge_{c\in A}(x \neq c)$. The notations $\exists^{\geqslant n} x \ldots$ and $\exists^{=n} x$____ will abbreviate the formulas stating that '*there exist at least n elements such that* ... *holds*' and '*there exist exactly n elements such that* ____ *holds*', respectively. If C is a finite set of constants, then the expression $C \subseteq P^\varepsilon$ means that $\bigwedge_{c\in C} P^\varepsilon(c)$. Let $n \in \omega$, $\varepsilon \in 2^m$, let C be a finite set of constants, and let E be an equivalence on C. In what follows, we define the sentences describing the structure of the partitionings of a model that are induced by P^ε:

$$\varphi^{=n}_{\varepsilon,E,C} \quad = \quad \exists^{=n} x \ (P^\varepsilon(x) \wedge x \notin C \wedge \eta_E \wedge C \subseteq P^\varepsilon)\,,$$

$$\varphi^{\geqslant n}_{\varepsilon,E,C} \quad = \quad \exists^{\geqslant n} x \ (P^\varepsilon(x) \wedge x \notin C \wedge \eta_E \wedge C \subseteq P^\varepsilon)\,.$$

For a simple signature σ, we call formulas of the form $\bigwedge_{\varepsilon\in 2^l}\Phi_\varepsilon$ *base,* if Φ_ε is a sentence of the form $\varphi^{=n}_{\varepsilon,E_\varepsilon,C_\varepsilon}$ or of the form $\varphi^{\geqslant n}_{\varepsilon,E_\varepsilon,C_\varepsilon}$ and the sets C_ε satisfy $C_{\varepsilon_0}\cap C_{\varepsilon_1}=\varnothing$ for $\varepsilon_0\neq\varepsilon_1$, $\{c_i \mid i<q\}=\bigcup_{\varepsilon\in 2^l} C_\varepsilon$, and for each $\varepsilon\in 2^l$, E_ε is an equivalences on C_ε.

The following lemma is rather straightforward:

Lemma 5 *Let φ and ψ be base formulas. Then their conjunction is either equivalent to a base formula or unsatisfiable.*

Let us introduce some additional notation. Let Δ be an arbitrary set of sentences. The notation $\mathfrak{M}(\Delta)\mathfrak{N}$ means that, for each $\varphi\in\Delta$, the condition $\mathfrak{M}\models\varphi$ yields $\mathfrak{N}\models\varphi$. We need the following:

Proposition 3 ([13, Lemma 3.2.1]) *Let Δ be a disjunction-closed set of sentences and let $\mathcal{T}$ be a consistent theory such that for all models $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M}\models\mathcal{T}$ and $\mathfrak{M}(\Delta)\mathfrak{N}$ then $\mathfrak{N}\models\ \mathcal{T}$. Then $\mathcal{T}$ is axiomatizable by sentences from Δ.*

Corollary 1 *Every consistent sentence in a simple signature is equivalent to a disjunction of base formulas.*

Proof. Take Δ in Proposition 3 to be the set of all disjunction-closed base formulas. Then for at most countable models $\mathfrak{M}$ and $\mathfrak{N}$, the condition $\mathfrak{M}(\Delta)\mathfrak{N}$ yields $\mathfrak{M}\cong\mathfrak{N}$. To complete the proof, it suffices to apply Proposition 3 and then Lemma 5. □

We proceed with the proof of Proposition 2. We are able now to define the formula φ^τ for $\tau\subseteq\sigma$. By Corollary 1, we can assume that each satisfiable formula is a disjunction of base formulas. Let us first define φ^τ for base formulas. Without loss of generality we can assume that

$$\tau=\left\langle (P_i)_{i<l'}\,;(c_i)_{i<q'}\right\rangle;\ \ l'\leqslant l,\ q'\leqslant q.$$

Let φ be a base formula in signature σ and

$$\varphi=\bigwedge_{\varepsilon\in 2^l}\varphi^{\lambda_\varepsilon n_\varepsilon}_{\varepsilon,E_\varepsilon,C_\varepsilon},\quad \lambda_\varepsilon\in\{=,\geqslant\},\ \text{for all } \varepsilon\in 2^l.$$

Given $\varepsilon\in 2^{l'}$, we consider all tuples γ such that $\varepsilon\subseteq\gamma\in 2^l$ and for every γ, we set m_γ equal to the number of equivalence classes E_γ that do not contain constants from $C'=\{c_i\mid i<q'\}$. Define the sentence ϕ_ε as:

$$\exists^{\lambda_\varepsilon m_\varepsilon}x\left[P^\varepsilon(x)\wedge\left(x\notin\bigcup_{\varepsilon\subseteq\gamma\in 2^l}C_\gamma\cap C'\right)\wedge\eta_{E_\varepsilon}\wedge\left(\bigcup_{\varepsilon\subseteq\gamma\in 2^l}C_\gamma\cap C'\subseteq P^\epsilon\right)\right],$$

where

$$\begin{aligned}
E_\varepsilon &= \bigcup_{\varepsilon\subseteq\gamma\in 2^l} E_\gamma,\\
m_\varepsilon &= \sum_{\varepsilon\subseteq\gamma\in 2^l} (n_\gamma + m_\gamma),\\
\lambda_\varepsilon &= \begin{cases} \geqslant & \text{if at least one of } \lambda_\gamma, \varepsilon\subseteq\gamma\in 2^l \text{ equals } \leqslant\\ = & \text{otherwise.}\end{cases}
\end{aligned}$$

Now set φ^τ equal to $\bigwedge_{\varepsilon\in 2^{l'}}\phi_\varepsilon$. If φ is a disjunction of base formulas θ_i, $i\in I$, then we set φ^τ equal to $\bigvee_{i\in I}\theta_i^\tau$.

Lemma 6 *1. $\varphi\vdash\varphi^\tau$;*

2. every model of φ^τ can be expanded to a model of φ;

3. if $\varphi\vdash\psi$ and $\psi\in L_\tau$ then $\varphi^\tau\vdash\psi$.

Proof. The first two properties are rather straightforward. To prove the third one, we assume that $\mathfrak{M}\models\varphi^\tau$ and demonstrate that $\mathfrak{M}\models\psi$. Let us expand $\mathfrak{M}$ to the model $\mathfrak{M}'\models\varphi$; we have $\mathfrak{M}'\models\psi$. By restricting this model to the signature τ, we obtain $\mathfrak{M}\models\psi$; hence, $\varphi^\tau\vdash\psi$. □

Note that we have described an effective procedure to compute φ^τ by a disjunction of base formulas. Given a sentence φ in a simple signature, we enumerate all possible consequences of φ and find a disjunction of base formulas equivalent to φ. After that, the above mentioned procedure gives the sentence φ^τ.

Finally, let us mention that the theory $\mathcal{T}$ of the class of all models of a simple signature σ is computably enumerable and decidable due to the following. There exists a computable sequence of theories $(\mathcal{T}_i)_{i<\omega}$ consisting of all complete extensions of $\mathcal{T}$. Indeed, to build a complete extension of $\mathcal{T}$, it suffices to formulate all equalities (and inequalities) between constants, their distribution between the predicates P^ε, and the number of elements (from 0 to ∞) different from constants in each P^ε. This can be easily formulated by base formulas and gives countably categorical theories. It suffices to apply the following result by Yu.L.Ershov:

Theorem 3 ([14]) *A theory $\mathcal{T}$ is decidable iff it is computably enumerable and there exists a computable sequence of complete theories $(T_i)_{i<\omega}$ such that $\mathcal{T}=\bigcap_{i<\omega}\mathcal{T}_i$.*

Proposition 2 is proved.

We would like to thank Prof. Dr. Karl-Erich Wolff for his hospitality during our visit to TU Darmstadt (Germany), where we have obtained a part of these results. We also thank all participants of the international RFBR-DFG project "COMO" and O.V. Kudinov for his valuable pieces of advice and comments on this paper.

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