On the Complexity of Semantic Integration of OWL Ontologies

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Abstract

We propose a new mechanism for integration of OWL ontologies using semantic import relations. In contrast to the standard OWL importing, we do not require all axioms of the imported ontologies to be taken into account for reasoning tasks, but only their logical implications over a chosen signature. This property comes natural in many ontology integration scenarios, especially when the number of ontologies is large. In this paper, we study the complexity of reasoning over ontologies with semantic import relations and establish a range of tight complexity bounds for various fragments of OWL.

1 Introduction and Motivation

Logic-based ontology languages such as OWL and OWL 2 [Cuenca Grau *et al.*, 2008] are becoming increasingly popular means for representation, integration, and querying of information, particularly in life sciences, such as Biology and Medicine. For example, a repository of Open Biological and Biomedical Ontologies [Smith *et al.*, 2007] is comprised of over eighty specialised ontologies on such diverse topics as molecular functions, biological processes, and cellular components. Ontology integration, in particular, aims at organizing information on different domains in a modular way so that information from one ontology can be reused in other ontologies. For example, the ontology of diseases may reference anatomical structures to describe the location of diseases, or genes with which the diseases are likely to be correlated.

Integration of multiple ontologies in OWL is organized via *importing*: an OWL ontology can refer to one or several other OWL ontologies, whose axioms must be implicitly present in the ontology. The importing mechanism is simple in that it does not require any significant modification of the underlying reasoning algorithms: in order to answer a query over an ontology with an import declaration, it is sufficient to apply the algorithm to the *import closure* consisting of the axioms of the ontology plus the axioms of the ontologies that are imported (possibly indirectly). For example, if ontology $\mathcal{O}_1$ imports ontologies $\mathcal{O}_2$ and $\mathcal{O}_3$, each of which, in turn, imports ontology $\mathcal{O}_4$, then the import closure of $\mathcal{O}_1$ consists of all axioms of $\mathcal{O}_1 - \mathcal{O}_4$. Provided these axioms altogether are expressible in the same fragment of OWL as each $\mathcal{O}_1 - \mathcal{O}_4$ is, a reasoning algorithm for this fragment can be used to answer queries over the import closure of $\mathcal{O}_1$. Then, since the size of such import closure is the same as the combined size of all ontologies involved, the computational complexity of reasoning over ontologies with imports remains the same as for ontologies without imports.

Although the OWL importing mechanism may work well for simple ontology integration scenarios, it may cause some undesirable side effects if used in complex import situations. To illustrate the problem, suppose that in the above example, $\mathcal{O}_4$ is an ontology describing a typical university. It may include concepts such as $\mathsf{Student}$, $\mathsf{Professor}$, Course, and axioms stating, e.g., that each professor must teach some course and that students are disjoint with professors:

$$\mathsf{Professor} \sqsubseteq \exists \mathsf{teaches}.\mathsf{Course}, \tag{1}$$
$$\mathsf{Student} \sqcap \mathsf{Professor} \sqsubseteq \bot. \tag{2}$$

Now suppose that $\mathcal{O}_2$ and $\mathcal{O}_3$ are ontologies describing respectively, Oxford and Cambridge universities that use $\mathcal{O}_4$ as a prototype. For example, $\mathcal{O}_2$ may include *mapping axioms*

$$\mathsf{OxfordStudent} \equiv \mathsf{Student}, \tag{3}$$
$$\mathsf{OxfordProfessor} \equiv \mathsf{Professor}, \tag{4}$$
$$\mathsf{OxfordCourse} \equiv \mathsf{Course}, \tag{5}$$

from which, due to (1), it is now possible to conclude that each Oxford professor must teach some Oxford course:

$$\mathsf{OxfordProfessor} \sqsubseteq \exists \mathsf{teaches}.\mathsf{OxfordCourse}. \tag{6}$$

Likewise, using similar mapping axioms in $\mathcal{O}_3$, it is possible to obtain that Cambridge students and professors are disjoint:

$$\mathsf{CambridgeStudent} \sqcap \mathsf{CambridgeProfessor} \sqsubseteq \bot. \tag{7}$$

Finally, suppose that $\mathcal{O}_1$ is an ontology aggregating information about UK universities, importing, among others, the ontologies $\mathcal{O}_2$ and $\mathcal{O}_3$ for Oxford and Cambridge universities.

Although the described scenario seems plausible, there will be some undesirable consequences in $\mathcal{O}_1$ due to the mapping axioms of $\mathcal{O}_2$ and $\mathcal{O}_3$ occurring in the import closure:

$$\mathsf{OxfordStudent} \equiv \mathsf{CambridgeStudent}, \tag{8}$$
$$\mathsf{OxfordProfessor} \equiv \mathsf{CambridgeProfessor}, \tag{9}$$
$$\mathsf{OxfordCourse} \equiv \mathsf{CambridgeCourse}. \tag{10}$$

The main reason for these consequences is that the ontologies $\mathcal{O}_2$ and $\mathcal{O}_3$ happen to reuse the same ontology $\mathcal{O}_4$ in two different and incompatible ways. Had they instead used two different 'copies' of $\mathcal{O}_4$ as prototypes (with concepts renamed apart), no such problem would take place. Arguably, the primary purpose of $\mathcal{O}_2$ and $\mathcal{O}_3$ is to provide semantic description of the vocabulary for Oxford and Cambridge universities, and the means of how it is achieved—either by writing the axioms directly or reusing third party ontologies such as $\mathcal{O}_4$—should be an internal matter of these two ontologies and should not be exposed to the ontologies that import them.

Motivated by the described scenario, in this paper we consider a refined mechanism for importing of OWL ontologies called *semantic importing*. The main difference with the standard OWL importing, is that each import is limited only to a subset of symbols. Intuitively, only logical properties of these symbols entailed by the imported ontology should be imported. These symbols can be regarded as the public (or external) vocabulary of the imported ontologies. For example, ontology $\mathcal{O}_2$ may declare the symbols $\mathsf{OxfordStudent}$, $\mathsf{OxfordProfessor}$, $\mathsf{OxfordCourse}$, and $\mathsf{teaches}$ public, leaving the remaining symbols only for the internal use.

The main results of this paper are tight complexity bounds for reasoning over ontologies with semantic imports. We consider ontologies formulated in different fragments of OWL starting from the propositional (role-free) Horn fragment $\mathcal{H}$, full propositional (role-free) fragment $\mathcal{P}$, and concluding with the Description Logic (DL) $\mathcal{SROIQ}$, which corresponds to OWL 2. We also distinguish the case of acyclic imports, when ontologies cannot (possibly indirectly) import themselves. Our completeness results for ranges of DLs are summarized in the following table, where $\mathfrak{a}$ and $\mathfrak{c}$ denote the case of acyclic/cyclic imports respectively:

DLs	Completeness	Theorems
$\mathcal{EL} - \mathcal{EL}^{++}$	$\mathsf{ExpTime}$ $\mathfrak{a}$	1, 10
containing $\mathcal{EL}$	RE (undecidable) $\mathfrak{c}$	2, 11
$\mathcal{ALC} - \mathcal{SHIQ}$	$\mathsf{2ExpTime}$ $\mathfrak{a}$	3, 10
$\mathcal{R} - \mathcal{SRIQ}$	$\mathsf{3ExpTime}$ $\mathfrak{a}$	4, 10
$\mathcal{ALCHOIF} - \mathcal{SHOIQ}$	$\mathsf{coN2ExpTime}$ $\mathfrak{a}$	5, 10
$\mathcal{ROIF} - \mathcal{SROIQ}$	$\mathsf{coN3ExpTime}$ $\mathfrak{a}$	6, 10
$\mathcal{H}$ - $\mathcal{P}$	$\mathsf{ExpTime}$ $\mathfrak{c}$	7, 12
$\mathcal{H}$ - $\mathcal{P}$	PSpace $\mathfrak{a}$	8, 13

The paper is organized as follows. In Sections 2-3 we describe related work and introduce basic notations. In Section 4 we formulate the problem of entailment in ontology networks. In Section 5 we prove results on the expressiveness of ontology networks and use them to show hardness of entailment in Section 6. Finally, in Section 7 we demonstrate that entailment in ontology networks reduces to standard entailment in Description Logics and use these results to prove in Section 8 that the obtained complexity bounds are tight.

2 Related Work

Frameworks for combining ontologies share the natural view that interpretations of linked ontologies must satisfy certain correspondence constraints. Most existing approaches (e.g. see an overview in [Homola and Serafini, 2010]) consider a model of a combination of ontologies as a tuple of interpretations, one for each ontology, with *correspondence relations* between the interpretation domains. These relations allow for propagation of semantics of entities (e.g., concepts, roles) from one ontology to another by providing interpretation for constructs stating links between ontology entities. The constructs are bridge rules in DDL [Borgida and Serafini, 2003], local/foreign symbol labels in PDL [Bao *et al.*, 2009] and the approach of [Pan *et al.*, 2006], link properties in $\mathcal{E}$-connections [Grau *et al.*, 2009], and alignment relations in [Euzenat *et al.*, 2007]. The last approach originates from the field of ontology matching [Shvaiko and Euzenat, 2013], which is a neighbour topic lying out of the scope of our paper, since it concentrates on computing matchings, but not on reasoning with them. The semantics for a combination of ontologies proposed in these approaches are in general not compatible with the conventional OWL importing mechanism. If an ontology $\mathcal{O}_1$ references some ontology $\mathcal{O}_2$, then correspondence relations guarantee propagation of certain entailments expressible in the language of $\mathcal{O}_2$ into the ontology $\mathcal{O}_1$. As a rule, the class of propagated entailments is not broad enough to simulate entailment form the union of ontologies, as required in OWL importing. The approach of [Grau and Motik, 2012] tries to bridge this gap by putting restrictions on combined ontologies, i.e. by considering conservative importing. Semantics given by tuples of interpretations may cause undesired effects, when combining two ontologies $\mathcal{O}_1, \mathcal{O}_2$ which both refine the same ontology $\mathcal{O}$ (cf. the motivating example from the introduction). Ontologies $\mathcal{O}_i$ may refine concepts of $\mathcal{O}$ in different ways, which may conflict to each other, while being consistent separately. The semantics given by tuples of interpretations makes supporting such integration scenarios problematic, since a single interpretation of $\mathcal{O}$ must be in correspondence with interpretations of both ontologies, $\mathcal{O}_1$ and $\mathcal{O}_2$. The integration mechanism proposed in this paper is conceptually simple. In order to reference an external ontology $\mathcal{O}$ from a local one, one has only to specify an import relation, which defines a set of symbols, whose semantics should be borrowed from $\mathcal{O}$. The symbols can be used freely in the axioms of the local ontology, no additional language constructs are required. This resembles the approaches [Bao *et al.*, 2009; Pan *et al.*, 2006], although theoretically, we do not distinguish between local and external symbols in ontologies (we note that this feature can be easily integrated). Importantly, in our approach every ontology has its own view on ontologies it refines and the views are independent between ontologies unless coordinated by the 'topology' of import relations.

3 Preliminaries

We assume that the reader is familiar with the family of Description Logics from $\mathcal{EL}$ to $\mathcal{SROIQ}$, for which the syntax is defined using a recursively enumerable alphabet consisting of infinite disjoint sets $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_i}$ of *concept names* (or *primitive concepts*), *roles*, and *nominals*, respectively. We also consider DLs $\mathcal{H}$ and $\mathcal{P}$, which are the role-free fragments of $\mathcal{EL}$ and $\mathcal{ALC}$, respectively. Thus, $\mathcal{P}$ corresponds to the classical propositional logic and $\mathcal{H}$ corresponds to the Horn fragment thereof.

The semantics of DLs is given by means of (first-order) interpretations. An *interpretation* $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$ consists of a non-empty set Δ, the *domain* of $\mathcal{I}$, and an *interpretation function* $\cdot^{\mathcal{I}}$, that assigns to each $A \in N_C$ a subset $A^{\mathcal{I}} \subseteq \Delta$, to each $r \in N_R$ a binary relation $r^{\mathcal{I}} \subseteq \Delta \times \Delta$, and to each $a \in \mathsf{N_i}$ an element of the domain Δ. An interpretation $\mathcal{I}$ *satisfies* a concept inclusion $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds. An ontology is a set of concept inclusions which are called ontology *axioms*. $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$, written $\mathcal{I} \models \mathcal{O}$, if $\mathcal{I}$ satisfies all axioms of $\mathcal{O}$. An ontology $\mathcal{O}$ *entails* a concept inclusion $C \sqsubseteq D$, in symbols $\mathcal{O} \models C \sqsubseteq D$, if every model of $\mathcal{O}$ satisfies $C \sqsubseteq D$. As usual, for concepts C, D, the equivalence $C \equiv D$ stands for the pair of concept inclusions $C \sqsubseteq D$ and $D \sqsubseteq C$.

A *signature* is a subset of $\mathsf{N_C} \cup \mathsf{N_R} \cup \mathsf{N_i}$. Interpretations $\mathcal{I}$ and $\mathcal{J}$ are said to *agree* on a signature Σ, written as $\mathcal{I} =_\Sigma \mathcal{J}$, if the domains of $\mathcal{I}$ and $\mathcal{J}$ coincide and the interpretation of Σ-symbols in $\mathcal{I}$ is the same as in $\mathcal{J}$. We denote the reduct of an interpretation $\mathcal{I}$ onto a signature Σ as $\mathcal{I}|_\Sigma$. The signature of a concept C, denoted as $\mathtt{sig}(C)$, is the set of all concept names, roles, and nominals occurring in C. The signature of a concept inclusion or an ontology is defined identically.

4 Semantic Importing

Given a signature Σ, suppose one wants to import into an ontology $\mathcal{O}_1$ the semantics of Σ-symbols defined by some other ontology $\mathcal{O}_2$, while ignoring the rest of the symbols from $\mathcal{O}_2$. Intuitively, importing the semantics of Σ-symbols means reducing the class of models of $\mathcal{O}_1$ by removing those models that violate the restrictions on interpretation of these symbols, which are imposed by the axioms of $\mathcal{O}_2$:

Definition 1. *A* (semantic) import relation *is a tuple* $\pi = \langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle$ *where* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are ontologies and* Σ *a signature. In this case, we say that* $\mathcal{O}_1$ imports Σ from $\mathcal{O}_2$. *We say that a model* $\mathcal{I} \models \mathcal{O}_1$ *satisfies the import relation* π *if there exists a model* $\mathcal{J} \models \mathcal{O}_2$ *such that* $\mathcal{I} =_\Sigma \mathcal{J}$.

Example 1. *Consider the import relation* $\pi = \langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle$, *with* $\mathcal{O}_1 = \{B \sqsubseteq C\}$, $\mathcal{O}_2 = \{A \sqsubseteq \exists r.B, \exists r.C \sqsubseteq D\}$, *and* $\Sigma = \{A, B, C, D\}$. *It can be easily shown using Definition 1 that a model* $\mathcal{I} \models \mathcal{O}_1$ *satisfies* π *if and only if* $\mathcal{I} \models A \sqsubseteq D$.

Note that if Σ contains all symbols in $\mathcal{O}_2$ then $\mathcal{I} \models \mathcal{O}_1$ satisfies $\pi = \langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle$ if and only if $\mathcal{I} \models \mathcal{O}_1 \cup \mathcal{O}_2$. That is, the standard OWL import relation is a special case of the semantic import relation, when the signature contains all the symbols from the imported ontology.

If $\mathcal{O}$ has several import relations $\phi_i = \langle \mathcal{O}, \Sigma_i, \mathcal{O}_i \rangle$, $(1 \leq i \leq n)$, one can define the entailment from $\mathcal{O}$ by considering only those models of $\mathcal{O}$ that satisfy all imports: $\mathcal{O} \models \alpha$ if $\mathcal{I} \models \alpha$ for every $\mathcal{I} \models \mathcal{O}$ which satisfies all $\pi_1, \ldots, \pi_n$. In practice, however, import relations can be nested: imported ontologies can themselves import other ontologies and so on. The following definition generalizes entailment to such situations.

Definition 2. *An* ontology network *is a finite set* $\mathcal{N}$ *of import relations between ontologies. For a DL* $\mathcal{L}$, *a* $\mathcal{L}$-ontology network *is a network, in which every ontology is a set of* $\mathcal{L}$-*axioms. A* model agreement *for* $\mathcal{N}$ *(over a domain* Δ*) is a mapping* μ *that assigns to every ontology* $\mathcal{O}$ *occurring in* $\mathcal{N}$ *a class* $\mu(\mathcal{O})$ *of models of* $\mathcal{O}$ *with domain* Δ *such that for every* $\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle \in \mathcal{N}$ *and every* $\mathcal{I}_1 \in \mu(\mathcal{O}_1)$ *there exists* $\mathcal{I}_2 \in \mu(\mathcal{O}_2)$ *such that* $\mathcal{I}_1 =_\Sigma \mathcal{I}_2$. *An interpretation* $\mathcal{I}$ *is a model of* $\mathcal{O}$ *in the network* $\mathcal{N}$ *(notation* $\mathcal{I} \models_\mathcal{N} \mathcal{O}$*) if there exists a model agreement* μ *for* $\mathcal{N}$ *such that* $\mathcal{I} \in \mu(\mathcal{O})$. *An ontology* $\mathcal{O}$ *entails a concept inclusion* φ *in the network* $\mathcal{N}$ *(notation* $\mathcal{O} \models_\mathcal{N} \varphi$*) if* $\mathcal{I} \models \varphi$, *whenever* $\mathcal{I} \models_\mathcal{N} \mathcal{O}$.

An ontology network can be seen as a labeled directed multigraph in which nodes are labeled by ontologies and edges are labeled by sets of signature symbols. Each edge in this graph, thus, represents an import relation between two ontologies. Note that Definition 2 also allows for *cyclic networks* if this graph is cyclic. That is, an ontology may refer to itself through a chain of import relations. Note that if $\mathcal{O} \models \varphi$ then $\mathcal{O} \models_\mathcal{N} \varphi$, for every network $\mathcal{N}$.

Example 2. *Consider the following (cyclic) network* $\mathcal{N} = \{\langle \mathcal{O}, \Sigma', \mathcal{O}' \rangle, \langle \mathcal{O}', \Sigma, \mathcal{O} \rangle\}$, *where*

- $\mathcal{O} = \{A \sqsubseteq B,\ A \equiv A',\ B \equiv B'\}$
- $\mathcal{O}' = \{A \equiv \exists r.A',\ B \equiv \exists r.B'\}$
- $\Sigma = \{A, B, r\}$, $\Sigma' = \{A', B', r\}$

Let μ *be any model agreement for* $\mathcal{N}$. *Since* $\langle \mathcal{O}', \Sigma, \mathcal{O} \rangle \in \mathcal{N}$, *by Definition 2, for every* $\mathcal{I}' \in \mu(\mathcal{O}')$ *there exists* $\mathcal{I} \in \mu(\mathcal{O})$ *such that* $\mathcal{I}' =_\Sigma \mathcal{I}$. *Since* $\mathcal{I} \models \mathcal{O} \models A \sqsubseteq B$ *and* $\{A, B\} \subseteq \Sigma$, *we have* $\mathcal{I}' \models A \sqsubseteq B$. *As* $\mathcal{I}' \models \mathcal{O}'$, *it also holds* $\mathcal{I}' \models \exists r.A' \sqsubseteq \exists r.B'$ *for every* $\mathcal{I}' \in \mu(\mathcal{O}')$ *and thus:*

$$\mathcal{O}' \models_\mathcal{N} \exists r.A' \sqsubseteq \exists r.B'. \tag{11}$$

Similarly, since $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$, *for every* $\mathcal{I} \in \mu(\mathcal{O})$ *there exists* $\mathcal{I}' \in \mu(\mathcal{O}')$ *such that* $\mathcal{I} =_{\Sigma'} \mathcal{I}'$. *Since* $\mathcal{I}' \models \exists r.A' \sqsubseteq \exists r.B'$ *and* $\{A', B', r\} \subseteq \Sigma'$, *we have* $\mathcal{I} \models \exists r.A' \sqsubseteq \exists r.B'$ *and since* $\mathcal{I} \models \mathcal{O}$, *it holds* $\mathcal{I} \models \exists r.A \sqsubseteq \exists r.B$, *for every* $\mathcal{I} \in \mu(\mathcal{O})$, *and hence:*

$$\mathcal{O} \models_\mathcal{N} \exists r.A \sqsubseteq \exists r.B. \tag{12}$$

By repeating these arguments we similarly obtain:

$$\mathcal{O}' \models_\mathcal{N} \exists r.\exists r.A' \sqsubseteq \exists r.\exists r.B', \tag{13}$$

$$\mathcal{O} \models_\mathcal{N} \exists r.\exists r.A \sqsubseteq \exists r.\exists r.B, \tag{14}$$

$$\mathcal{O}' \models_\mathcal{N} \exists r.\exists r.\exists r.A' \sqsubseteq \exists r.\exists r.\exists r.B', \tag{15}$$

$$\mathcal{O} \models_\mathcal{N} \exists r.\exists r.\exists r.A \sqsubseteq \exists r.\exists r.\exists r.B, \tag{16}$$

and so on. Note the matching nestings of $\exists r$ *in these axioms.*

In this paper, we are concerned with the complexity of *entailment in ontology networks*, that is, given a network $\mathcal{N}$, an ontology $\mathcal{O}$ and an axiom φ, decide whether $\mathcal{O} \models_\mathcal{N} \varphi$. We study the complexity of this problem wrt the *size* of an ontology network $\mathcal{N}$, which is defined as the total length of axioms (considered as strings) occurring in ontologies from $\mathcal{N}$.

5 Expresiveness of Ontology Networks

We illustrate the expressiveness of ontology networks by showing that acyclic networks allow for succinctly representing axioms with nested concepts and role chains of exponential size, while cyclic ones allow for succinctly representing infinite sets of axioms of a special form.

For a natural number $n \geqslant 0$, let $\exists(r, C)^n.D$ be a shortcut for the nested concept

$$\underbrace{\exists r.(C \sqcap \exists r.(C \sqcap \cdots \sqcap \exists r.(C}_{n \text{ times}} \sqcap D) \cdots)) \tag{17}$$

where C, D are DL concepts and r a role (in case $n = 0$ the above concept is set to be D). For $n \geqslant 1$, let $(r)^n$ denote the role chain

$$\underbrace{r \circ \ldots \circ r}_{n \text{ times}} \tag{18}$$

For a given $n \geqslant 0$, let $\mathtt{1exp}(n)$ be the notation for 2^n and for $k \geqslant 1$, let $(\mathtt{k+1})\mathtt{exp}(n) = 2^{\mathtt{kexp}(n)}$. Then $\exists(r, C)^{\mathtt{kexp}(n)}.D$ (respectively, $(r)^{\mathtt{kexp}(n)}$) stands for a nested concept (role chain) of the form above having size exponential in n.

In the following, we use abbreviations $\exists(r, C)^n := \exists(r, C)^n.\top$ and $\exists r^n.C := \exists(r, \top)^n.C$. For roles r, s and $n \geqslant 2$, let $(r)^{<n} \sqsubseteq s$ be an abbreviation for the set of role inclusions $\{(r)^k \sqsubseteq s \mid 1 \leqslant k < n\}$. For $n \geqslant 1$, the expression $\forall r^n.C$ will be used as a shortcut for $\neg\exists r.\exists r^{n-1}.\neg C$ and for $n \geqslant 2$, $\forall r^{<n}.C$ will stand for $\bigsqcap_{1 \leqslant m < n} \forall r^m.C$.

Let $\mathcal{O}$ be an ontology and $\mathcal{N}$ an ontology network. $\mathcal{O}$ is said to be *expressible* by $\mathcal{N}$ if there is an ontology $\mathcal{O}_\mathcal{N}$ in $\mathcal{N}$ such that $\{\mathcal{I}|_{\mathtt{sig}(\mathcal{O})} \mid \mathcal{I} \models_\mathcal{N} \mathcal{O}_\mathcal{N}\} = \{\mathcal{J}|_{\mathtt{sig}(\mathcal{O})} \mid \mathcal{J} \models \mathcal{O}\}$. In other words, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} \mathcal{O}$ and any model $\mathcal{J} \models \mathcal{O}$ can be expanded to a model $\mathcal{I} \models_\mathcal{N} \mathcal{O}_\mathcal{N}$. Note that this yields that $\mathcal{O} \models \varphi$ iff $\mathcal{O}_\mathcal{N} \models_\mathcal{N} \varphi$, for any concept inclusion φ such that $\mathtt{sig}(\varphi) \subseteq \mathtt{sig}(\mathcal{O})$. In case we want to stress the role of ontology $\mathcal{O}_\mathcal{N}$ in the network $\mathcal{N}$, we say that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-*expressible*. An axiom φ is expressible by a network $\mathcal{N}$ if so is ontology $\mathcal{O} = \{\varphi\}$.

The next two lemmas follow immediately from the definition of expressibility.

Lemma 1. *Every ontology $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O})$-expressible, where $\mathcal{N}$ is the network consisting of the single import relation $\langle \mathcal{O}, \varnothing, \varnothing \rangle$. If an ontology $\mathcal{O}_i$ is $(\mathcal{N}_i, \mathcal{O}'_i)$-expressible, for a network $\mathcal{N}_i$, ontology $\mathcal{O}'_i$, and $i = 1, 2$, then $\mathcal{O}_1 \cup \mathcal{O}_2$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible, for ontology $\mathcal{O}_\mathcal{N} = \varnothing$ and a network*[1] *$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \{\langle \mathcal{O}_\mathcal{N}, \mathtt{sig}(\mathcal{O}_i), \mathcal{O}'_i \rangle\}_{i=1,2}$.*

For an axiom φ and a set of concepts $\{C_1, \ldots, C_n\}$, $n \geqslant 1$, let us denote by $\varphi[C_1 \mapsto D_1, \ldots, C_n \mapsto D_n]$ the axiom obtained by substituting every concept C_i with a concept D_i in φ. For an ontology $\mathcal{O}$, let $\mathcal{O}[C_1 \mapsto D_1, \ldots, C_n \mapsto D_n]$ be a notation for $\bigcup_{\varphi \in \mathcal{O}} \varphi[C_1 \mapsto D_1, \ldots, C_n \mapsto D_n]$.

Lemma 2. *Let $\mathcal{L}$ be a DL and $\mathcal{O}$ an ontology, which is expressible by a $\mathcal{L}$-ontology network $\mathcal{N}$. Let $C_1, \ldots, C_n$ be $\mathcal{L}$-concepts and $\{A_1, \ldots, A_n\}$ a set of concept names such that $A_i \in \mathtt{sig}(\mathcal{O})$, for $i = 1, \ldots, n$ and $n \geqslant 1$. Then ontology $\tilde{\mathcal{O}} = \mathcal{O}[A_1 \mapsto C_1, \ldots, A_n \mapsto C_n]$ is expressible by a $\mathcal{L}$-ontology network, which is acyclic if so is $\mathcal{N}$ and has size polynomial in the size of $\mathcal{N}$ and C_i, $i = 1, \ldots, n$.*

Proof. Denote $\Sigma = \{A_1, \ldots, A_n\}$ and let $\Sigma' = \{A'_1, \ldots, A'_n\}$ be a set of fresh concept names. Consider ontology $\mathcal{O}' = \mathcal{O} \cup \{A_i \equiv A'_i\}_{i=1,\ldots,n}$. By Lemma 1, $\mathcal{O}'$ is $(\mathcal{N}', \mathcal{O}_{\mathcal{N}'})$-expressible, for an ontology $\mathcal{O}_{\mathcal{N}'}$ and an acyclic $\mathcal{L}$-ontology network $\mathcal{N}'$ having a linear size (in the size of $\mathcal{N}$). Consider ontology network $\mathcal{N}'' = \langle \mathcal{O}_{\mathcal{N}''}, (\mathtt{sig}(\mathcal{O}') \setminus \Sigma) \cup \Sigma', \mathcal{O}_{\mathcal{N}'} \rangle$, where $\mathcal{O}_{\mathcal{N}''} = \varnothing$. Then obviously, ontology $\mathcal{O}'' = \mathcal{O}[A_1 \mapsto A'_1, \ldots, A_n \mapsto A'_n]$ is $(\mathcal{N}'', \mathcal{O}_{\mathcal{N}''})$-expressible. Similarly, by Lemma 1, ontology $\mathcal{O}''_C = \mathcal{O}'' \cup \{A'_i \equiv C_i\}_{i=1,\ldots,n}$ is $(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$-expressible, for an ontology $\mathcal{O}_{\tilde{\mathcal{N}}}$ and an acyclic $\mathcal{L}$-ontology network $\tilde{\mathcal{N}}$ having a linear size (in the size of $\mathcal{N}$ and C_i, for $i = 1, \ldots, n$). Clearly, it holds $\mathcal{O}_{\tilde{\mathcal{N}}} \models_{\tilde{\mathcal{N}}} \tilde{\mathcal{O}}$. On the other hand, any model $\mathcal{I} \models \tilde{\mathcal{O}}$ can be expanded to a model $\mathcal{J}$ of ontology $\mathcal{O}''_C$ by setting $(A'_i)^\mathcal{J} = (C_i)^\mathcal{I}$, for $i = 1, \ldots, n$. Since $\mathcal{O}''_C$ is $(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$-expressible, it follows that $\mathcal{I}$ can be expanded to a model $\tilde{J} \models_{\tilde{\mathcal{N}}} \mathcal{O}_{\tilde{\mathcal{N}}}$ and therefore, $\tilde{\mathcal{O}}$ is $(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$-expressible. $\square$

Now we proceed to the results on the succinct representation of exponentially long axioms and infinite sets of axioms of a special form. We begin with lemmas on the expressiveness of acyclic $\mathcal{EL}$- and $\mathcal{ALC}$-ontology networks.

Lemma 3. *An axiom φ of the form $Z \equiv \exists(r, A)^{\mathtt{1exp}(n)}.B$, where $Z, A, B \in \mathsf{N_C}$, and $n \geqslant 0$, is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in n.*

Proof. We prove by induction on n that there exists an acyclic $\mathcal{EL}$-ontology network $\mathcal{N}_n$ and ontology $\mathcal{O}_n$ such that φ is $(\mathcal{N}_n, \mathcal{O}_n)$-expressible. For $n = 0$, we define $\mathcal{N}_0 = \{\langle \mathcal{O}_0, \varnothing, \varnothing \rangle\}$ and $\mathcal{O}_0 = \{Z \equiv \exists r.(A \sqcap B)\}$.

In the induction step, let $\{Z \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.B\}$ be $(\mathcal{N}_{n-1}, \mathcal{O}_{n-1})$-expressible, for $n \geqslant 1$. Consider ontologies:

$$\mathcal{O}^1_{copy} = \{B \equiv U\} \tag{19}$$

$$\mathcal{O}^2_{copy} = \{U \equiv Z\} \tag{20}$$

$$\mathcal{O}_n = \varnothing \tag{21}$$

Let $\mathcal{N}_n$ be the union of $\mathcal{N}_{n-1}$ with the set of the following import relations: $\langle \mathcal{O}^i_{copy}, \{Z, A, B, r\}, \mathcal{O}_{n-1} \rangle$, for $i = 1, 2$, $\langle \mathcal{O}_n, \{Z, A, U, r\}, \mathcal{O}^1_{copy} \rangle$, and $\langle \mathcal{O}_n, \{U, A, B, r\}, \mathcal{O}^2_{copy} \rangle$.

Let us verify that $\{\mathcal{I}|_{\mathtt{sig}(\varphi)} \mid \mathcal{I} \models_{\mathcal{N}_n} \mathcal{O}_n\} = \{\mathcal{I}|_{\mathtt{sig}(\varphi)} \mid \mathcal{I} \models \varphi\}$. By the induction assumption we have $\mathcal{O}_{n-1} \models_{\mathcal{N}_n} Z \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.B$. Then by the definition of $\mathcal{N}_n$, it holds $\mathcal{O}^1_{copy} \models_{\mathcal{N}_n} Z \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.U$ and $\mathcal{O}^2_{copy} \models_{\mathcal{N}_n} U \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.B$ and thus, $\mathcal{O}_n \models_{\mathcal{N}_n} \{Z \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.U, U \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.B\}$, which yields $\mathcal{O}_n \models_{\mathcal{N}_n} \varphi$.

We now show that any model $\mathcal{I} \models \varphi$ can be expanded to a model $\mathcal{J}_n \models_{\mathcal{N}_n} \mathcal{O}_n$. Let us define $\mathcal{J}_n$ as an expansion of $\mathcal{I}$ by setting $U^{\mathcal{J}_n} = (\exists(r, A)^{\mathtt{1exp}(n-1)}.B)^\mathcal{I}$. Clearly, it holds $\mathcal{J}_n \models \mathcal{O}_n$ and there exists a model $\mathcal{I}_2 \models \mathcal{O}^2_{copy}$ such that $\mathcal{I}_2 =_{\{U,A,B,r\}} \mathcal{J}_n$ and $Z^{\mathcal{I}_2} = U^{\mathcal{I}_2}$. We have $\mathcal{I}_2 \models Z \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.B$ and hence by the induction assumption, there is a model $\mathcal{J}^2_{n-1} \models_{\mathcal{N}_n} \mathcal{O}_{n-1}$ such that $\mathcal{I}_2 =_{\{Z,A,B,r\}} \mathcal{J}^2_{n-1}$. Similarly, there is a model $\mathcal{I}_1 \models \mathcal{O}^1_{copy}$ such that $\mathcal{I}_1 =_{\{Z,A,U,r\}} \mathcal{J}_n$ and $B^{\mathcal{I}_1} = (\exists(r, A)^{\mathtt{1exp}(n-1)})^{\mathcal{I}_1}$. We have $\mathcal{I}_1 \models Z \equiv \exists(r, A)^{\mathtt{1exp}(n-1)}.B$, thus by the induction assumption, there is a model $\mathcal{J}^1_{n-1} \models_{\mathcal{N}_n} \mathcal{O}_{n-1}$ such that $\mathcal{I}_1 =_{\{Z,A,B,r\}} \mathcal{J}^1_{n-1}$. It follows that there exists a model

[1] Note that the size of $\mathcal{N}$ is linear in the sizes of $\mathcal{N}_1$ and $\mathcal{N}_2$.

agreement μ for $\mathcal{N}_n$ such that $\mu(\mathcal{O}_{n-1}) = \{\mathcal{J}^1_{n-1}, \mathcal{J}^2_{n-1}\}$, $\mu(\mathcal{O}^i_{copy}) = \{\mathcal{I}_i\}$, for $i = 1, 2$, and $\mu(\mathcal{O}_n) = \{\mathcal{J}_n\}$, which means that $\mathcal{J}_n \models_{\mathcal{N}_n} \mathcal{O}_n$. $\square$

The following two statements are proved identically to Lemma 3:

Lemma 4. *An axiom of the form $Z \sqsubseteq \exists(r, A)^{\mathtt{1exp}(n)}.B$, where $Z, A, B \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in n.*

Lemma 5. *An axiom of the form $Z \equiv \forall r^{\mathtt{1exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n.*

Next, we demonstrate the expressiveness of acyclic $\mathcal{R}$-ontology networks.

Lemma 6. *An axiom φ of the form $(r)^{\mathtt{2exp}(n)} \sqsubseteq s$, where r, s are roles and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. We use the idea of the proof of Lemma 3 and show by induction on n that there exists an acyclic $\mathcal{R}$-ontology network $\mathcal{N}_n$ and ontology $\mathcal{O}_n \in \mathcal{N}_n$ such that φ is $(\mathcal{N}_n, \mathcal{O}_n)$-expressible. For $n = 0$, we define $\mathcal{N}_0 = \{\langle \mathcal{O}_0, \varnothing, \varnothing\rangle\}$ and $\mathcal{O}_0 = \{r \circ r \sqsubseteq s\}$.

In the induction step, let $(r)^{\mathtt{2exp}(n-1)} \sqsubseteq s$ be $(\mathcal{N}_{n-1}, \mathcal{O}_{n-1})$-expressible, for $n \geqslant 1$. Consider ontologies:

$$\mathcal{O}^1_{copy} = \{s \sqsubseteq u\} \tag{22}$$

$$\mathcal{O}^2_{copy} = \{u \sqsubseteq r\} \tag{23}$$

$$\mathcal{O}_n = \varnothing \tag{24}$$

Let $\mathcal{N}_n$ be the union of $\mathcal{N}_{n-1}$ with the set of the following import relations: $\langle \mathcal{O}^i_{copy}, \{r, s\}, \mathcal{O}_{n-1}\rangle$, for $i = 1, 2$, $\langle \mathcal{O}_n, \{r, u\}, \mathcal{O}^1_{copy}\rangle$, and $\langle \mathcal{O}_n, \{u, s\}, \mathcal{O}^2_{copy}\rangle$.

We show that $\{\mathcal{I}|_{\mathtt{sig}(\varphi)} \mid \mathcal{I} \models_{\mathcal{N}_n} \mathcal{O}_n\} = \{\mathcal{I}|_{\mathtt{sig}(\varphi)} \mid \mathcal{I} \models \varphi\}$. By the induction assumption, we have $\mathcal{O}_{n-1} \models_{\mathcal{N}_n} (r)^{\mathtt{2exp}(n-1)} \sqsubseteq s$. Hence by the definition of $\mathcal{N}_n$, it holds $\mathcal{O}^1_{copy} \models_{\mathcal{N}_n} (r)^{\mathtt{2exp}(n-1)} \sqsubseteq u$ and $\mathcal{O}^1_{copy} \models_{\mathcal{N}_n} (u)^{\mathtt{2exp}(n-1)} \sqsubseteq s$ and therefore, $\mathcal{O}_n \models_{\mathcal{N}_n} \{(r)^{\mathtt{2exp}(n-1)} \sqsubseteq u, (u)^{\mathtt{2exp}(n-1)} \sqsubseteq s\}$, which means that $\mathcal{O}_n \models_{\mathcal{N}_n} \varphi$. By using an argument like in the proof of Lemma 3 one can verify that any model $\mathcal{I} \models \varphi$ can be expanded to a model $\mathcal{J}_n \models_{\mathcal{N}_n} \mathcal{O}_n$ by setting $u^{\mathcal{J}_n} = ((r)^{\mathtt{2exp}(n-1)})^{\mathcal{I}}$. $\square$

Lemma 7. *An ontology $\mathcal{O}$ given by the set of axioms $(r)^{<\mathtt{2exp}(n)} \sqsubseteq s$, where r, s are roles and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. We use a modification of the proof of Lemma 6 and show by induction on n that there exists an acyclic $\mathcal{R}$-ontology network $\mathcal{N}_n$ and ontology $\mathcal{O}_n \in \mathcal{N}_n$ such that $\mathcal{O}$ is $(\mathcal{N}_n, \mathcal{O}_n)$-expressible. For $n = 0$, we define $\mathcal{N}_0 = \{\langle \mathcal{O}_0, \varnothing, \varnothing\rangle\}$ and $\mathcal{O}_0 = \{r \sqsubseteq s\}$.

In the induction step, suppose ontology $\{(r)^m \sqsubseteq s \mid 1 \leqslant m \leqslant \mathtt{2exp}(n-1) - 1\}$ is $(\mathcal{N}_{n-1}, \mathcal{O}_{n-1})$-expressible, for $n \geqslant 1$. Consider ontologies:

$$\mathcal{O}^1_{copy} = \{s \sqsubseteq u_1\} \tag{25}$$

$$\mathcal{O}^2_{copy} = \{u_1 \sqsubseteq r, s \sqsubseteq u_2\} \tag{26}$$

$$\mathcal{O}_n = \{u_1 \sqsubseteq u_3,\ u_1 \circ u_1 \sqsubseteq u_3,\ u_2 \circ u_3 \sqsubseteq s,\ r \sqsubseteq s\} \tag{27}$$

Let $\mathcal{N}_n$ be the union of $\mathcal{N}_{n-1}$ with the set of the following import relations: $\langle \mathcal{O}^i_{copy}, \{r, s\}, \mathcal{O}_{n-1}\rangle$, for $i = 1, 2$, $\langle \mathcal{O}_n, \{r, u_1\}, \mathcal{O}^1_{copy}\rangle$, and $\langle \mathcal{O}_n, \{u_1, u_2\}, \mathcal{O}^2_{copy}\rangle$.

We show that $\{\mathcal{I}|_{\mathtt{sig}(\mathcal{O})} \mid \mathcal{I} \models_{\mathcal{N}_n} \mathcal{O}_n\} = \{\mathcal{I}|_{\mathtt{sig}(\mathcal{O})} \mid \mathcal{I} \models \mathcal{O}\}$. By the induction assumption, we have $\mathcal{O}_{n-1} \models_{\mathcal{N}_n} (r)^{<\mathtt{2exp}(n-1)} \sqsubseteq s$ and hence, by the definition of $\mathcal{N}_n$, it holds $\mathcal{O}_n \models_{\mathcal{N}_n} (r)^{<\mathtt{2exp}(n-1)} \sqsubseteq u_1$ and $\mathcal{O}_n \models_{\mathcal{N}_n} (u_1)^{<\mathtt{2exp}(n-1)} \sqsubseteq u_2$. Then $\mathcal{O}_n \models_{\mathcal{N}_n} (r)^m \sqsubseteq u_2$, for $1 \leqslant m \leqslant (\mathtt{2exp}(n-1) - 1)^2$.

Since $\mathcal{O}_n \models_{\mathcal{N}_n} (r)^{<\mathtt{2exp}(n-1)} \sqsubseteq u_1$ and $\{u_1 \sqsubseteq u_3,\ u_1 \circ u_1 \sqsubseteq u_3\} \subseteq \mathcal{O}_n$, it holds $\mathcal{O}_n \models_{\mathcal{N}_n} (r)^m \sqsubseteq u_3$, for $1 \leqslant m \leqslant 2(\mathtt{2exp}(n-1) - 1)$. Therefore, since $u_2 \circ u_3 \sqsubseteq s \in \mathcal{O}_n$, we obtain $\mathcal{O}_n \models_{\mathcal{N}_n} \{(r)^k \sqsubseteq s \mid 2 \leqslant k \leqslant m\}$, for $m = (\mathtt{2exp}(n-1) - 1)^2 + 2(\mathtt{2exp}(n-1) - 1) = \mathtt{2exp}(n) - 1$. Since $r \sqsubseteq s \in \mathcal{O}_n$, we conclude that $\mathcal{O}_n \models_{\mathcal{N}_n} (r)^{<\mathtt{2exp}(n)} \sqsubseteq s$.

By using an argument like in the proof of Lemma 3 one can verify that any model $\mathcal{I} \models (r)^{<\mathtt{2exp}(n)} \sqsubseteq s$ can be expanded to a model $\mathcal{J} \models_{\mathcal{N}_n} \mathcal{O}_n$ by setting $u_i^{\mathcal{J}} = \bigcup_{1 \leqslant k \leqslant m_i} ((r)^k)^{\mathcal{I}}$, for $i = 1, 2, 3$, where $m_1 = \mathtt{2exp}(n-1) - 1$, $m_2 = (\mathtt{2exp}(n-1) - 1)^2$, and $m_3 = 2(\mathtt{2exp}(n-1) - 1)$. $\square$

Lemma 8. *An axiom φ of the form $Z \sqsubseteq \forall r^{\mathtt{2exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. Consider ontology $\mathcal{O}$ consisting of axioms

$$Z \sqsubseteq \forall s.A, \quad (r)^{\mathtt{2exp}(n)} \sqsubseteq s \tag{28}$$

Clearly, $\mathcal{O} \models \varphi$ and any model $\mathcal{I} \models \varphi$ can be expanded to a model $\mathcal{J} \models \mathcal{O}$ by setting $s^{\mathcal{J}} = ((r)^{\mathtt{2exp}(n)})^{\mathcal{I}}$. By Lemmas 1, 6, $\mathcal{O}$ is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n, from which the claim follows. $\square$

The following statement is a direct consequence of Lemmas 1 and 7 and is proved identically to Lemma 8:

Lemma 9. *An axiom of the form $Z \sqsubseteq \forall r^{<\mathtt{2exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Now we are ready to prove the next statement, which is an analogue of Lemma 4 for the case of double exponent.

Lemma 10. *An axiom φ of the form $Z \sqsubseteq \exists(r, A)^{\mathtt{2exp}(n)}.B$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. Consider ontology $\bar{\mathcal{O}}$ consisting of axioms

$$Z \sqsubseteq \exists s.\top, \quad Z \sqsubseteq \forall s^{<\mathtt{2exp}(n)}.X, \quad Z \sqsubseteq \forall s^{\mathtt{2exp}(n)}.Y$$

$$s \sqsubseteq r$$

By Lemmas 1, 8, 9, $\bar{\mathcal{O}}$ is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n. Then by Lemma 2, so is ontology $\mathcal{O} = \bar{\mathcal{O}}[X \mapsto A \sqcap \exists s.\top,\ Y \mapsto A \sqcap B]$.

Clearly, we have $\mathcal{O} \models \varphi$. Now let $\mathcal{I}$ be an arbitrary model of φ and for $m = 2\texttt{exp}(n)$, let $x_0, \ldots, x_m$ be arbitrary domain elements such that $x_0 \in Z^{\mathcal{I}}$, $\langle x_0, x_1 \rangle \in r^{\mathcal{I}}$, and $\langle x_i, x_{i+1} \rangle \in r^{\mathcal{I}}$, $x_i \in A^{\mathcal{I}}$, for $1 \leqslant i < m$, and $x_m \in A^{\mathcal{I}} \sqcap B^{\mathcal{I}}$. Let $\mathcal{J}$ be an expansion of $\mathcal{I}$ in which $s^{\mathcal{J}} = \{\langle x_i, x_{i+1} \rangle\}_{0 \leqslant i < m}$. Then we have $\mathcal{J} \models \mathcal{O}$, from which the claim follows. □

Lemma 11. *An axiom φ of the form $Z \equiv \forall r^{2\texttt{exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. Consider ontology $\bar{\mathcal{O}}$ consisting of axioms

$$Z \sqsubseteq \forall r^{2\texttt{exp}(n)}.A, \quad \bar{Z} \sqsubseteq \exists r^{2\texttt{exp}(n)}.\bar{A}$$

By Lemmas 1, 8, 10, $\bar{\mathcal{O}}$ is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n and by Lemma 2, so is ontology $\mathcal{O} = \bar{\mathcal{O}}[\bar{Z} \mapsto \neg Z,\ \bar{A} \mapsto \neg A]$. It remains to note that $\mathcal{O}$ and $\{\varphi\}$ are equivalent, so the claim is proved. □

Lemma 12. *Let $\mathcal{L}$ be a DL and $\mathcal{O}$ an ontology, which is expressible by a $\mathcal{L}$-ontology network $\mathcal{N}$. Let $C_1, \ldots, C_m$ be $\mathcal{L}$-concepts and $\{A_1, \ldots, A_m\}$ a set of concept names such that $A_i \in \texttt{sig}(\mathcal{O})$, for $i = 1, \ldots, m$ and $m \geqslant 1$. Then for $k = 1, 2$ and $n \geqslant 0$, ontology $\tilde{\mathcal{O}} = \mathcal{O}[A_1 \mapsto \forall r^{\texttt{kexp}(n)}.C_1, \ldots, A_m \mapsto \forall r^{\texttt{kexp}(n)}.C_m]$ is expressible by a $\mathcal{L}'$-ontology network, which is acyclic if so is $\mathcal{N}$ and has size polynomial in the size of $\mathcal{N}$, n, and C_i, for $i = 1, \ldots, m$, where:*

- *$\mathcal{L}' = \mathcal{L}$ if $\mathcal{L}$ contains $\mathcal{ALC}$ and $k = 1$;*
- *$\mathcal{L}' = \mathcal{L}$ if $\mathcal{L}$ contains $\mathcal{R}$ and $k = 2$.*

Proof. The proof uses Lemmas 5, 11 and is identical to the proof of Lemma 2. □

The next statement is shown similarly by using Lemma 3:

Lemma 13. *In the conditions of Lemma 12, for $n \geqslant 0$, ontology $\tilde{\mathcal{O}} = \mathcal{O}[A_1 \mapsto \exists r^{1\texttt{exp}(n)}.C_1, \ldots, A_m \mapsto \exists r^{1\texttt{exp}(n)}.C_m]$ is expressible by a $\mathcal{L}'$-ontology network, which is acyclic if so is $\mathcal{N}$ and has size polynomial in the size of $\mathcal{N}$, n, and C_i, for $i = 1, \ldots, m$, where $\mathcal{L}' = \mathcal{L}$ if $\mathcal{L}$ contains $\mathcal{EL}$.*

Lemma 14. *Let $\mathcal{L}$ be a DL and $\mathcal{O}$ an ontology, which is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible, for a $\mathcal{L}$-ontology network $\mathcal{N}$ and an ontology $\mathcal{O}_{\mathcal{N}}$. Let $\{A_1, \ldots, A_n\}$, $n \geqslant 1$, be concept names such that $A_i \in \texttt{sig}(\mathcal{O})$, for $i = 1, \ldots, n$, and let $\{C_1, \ldots, C_n\}$ be $\mathcal{L}$-concepts, where every C_i is of the form $\exists(r, D)^p.A_i$, for some role r, concept name D, and $p \geqslant 1$. Then ontology $\tilde{\mathcal{O}} = \bigcup_{m \geqslant 0} \mathcal{O}_m$, where $\mathcal{O}_0 = \mathcal{O}$ and $\mathcal{O}_{m+1} = \mathcal{O}_m[A_1 \mapsto C_1, \ldots, A_n \mapsto C_n]$, for all $m \geqslant 0$, is expressible by a cyclic $\mathcal{L}$-ontology network.*

Proof. Let $\sigma = \{B_1, \ldots, B_k\} = \bigcup_{i=1,\ldots,n} (\texttt{sig}(C_i) \cap \mathsf{N_C})$ and $\sigma' = \{B'_1, \ldots, B'_k\}$ be a set of fresh concept names, which is disjoint with σ and $\texttt{sig}(\mathcal{O})$. Let $\{C'_1, \ldots, C'_n\}$ be 'copy' concepts obtained from $C_1, \ldots, C_n$ by replacing every B_i with B'_i, for $i = 1, \ldots, k$. Consider ontologies

$$\tilde{\mathcal{O}}_{\mathcal{N}'} = \{\ B_i \equiv B'_i\ \}_{i=1,\ldots,k}$$
$$\mathcal{O}' = \{\ A_i \equiv C'_i\ \}_{i=1,\ldots,n}$$

and an ontology network $\mathcal{N}'$ given by the union of $\mathcal{N}$ with the set of import relations

$$\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \texttt{sig}(\mathcal{O}), \mathcal{O}_{\mathcal{N}} \rangle,\ \langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \Sigma', \mathcal{O}' \rangle,\ \langle \mathcal{O}', \Sigma, \tilde{\mathcal{O}}_{\mathcal{N}'} \rangle$$

where $\Sigma = \texttt{sig}(\mathcal{O}) \cup \bigcup_{i=1,\ldots,n} \texttt{sig}(C_i)$ and $\Sigma' = (\Sigma \setminus \sigma) \cup \sigma'$. We claim that ontology $\tilde{\mathcal{O}}$ is $(\mathcal{N}', \tilde{\mathcal{O}}_{\mathcal{N}'})$-expressible. Denote $\tilde{\mathcal{O}}' = \bigcup_{m \geqslant 0} \mathcal{O}'_m$, where $\mathcal{O}'_m = \mathcal{O}_m[B_1 \mapsto B'_1, \ldots, B_k \mapsto B'_k]$, for all $m \geqslant 0$.

First, we show by induction that $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_m$, for all $m \geqslant 0$. The induction base for $n = 0$ is trivial, since we have $\mathcal{O}_0 = \mathcal{O}$ and $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible, $\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \texttt{sig}(\mathcal{O}), \mathcal{O}_{\mathcal{N}} \rangle \in \mathcal{N}'$, and thus, $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}$. Suppose $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_m$, for some $m \geqslant 0$. Since $\langle \mathcal{O}', \Sigma, \tilde{\mathcal{O}}_{\mathcal{N}'} \rangle \in \mathcal{N}'$, we have $\mathcal{O}' \models_{\mathcal{N}'} \mathcal{O}_m$ and thus, by the equivalences in $\mathcal{O}'$, it holds $\mathcal{O}' \models_{\mathcal{N}'} \mathcal{O}'_{m+1}$. Since $\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \Sigma', \mathcal{O}' \rangle$, we have $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}'_{m+1}$ and hence, by the equivalences in $\tilde{\mathcal{O}}_{\mathcal{N}'}$, it holds that $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_{m+1}$.

Now let $\mathcal{I}$ be an arbitrary model of ontology $\tilde{\mathcal{O}}$ and $\mathcal{I}_1$ be an expansion of $\mathcal{I}$, in which every B'_i is interpreted as $(B_i)^{\mathcal{I}}$, for $i = 1, \ldots, k$. Clearly, it holds $\mathcal{I}_1 \models \tilde{\mathcal{O}}_{\mathcal{N}'}$ and thus, we have $\mathcal{I}_1 \models \tilde{\mathcal{O}}_{\mathcal{N}'} \cup \tilde{\mathcal{O}}$. We show that $\mathcal{I}_1 \models_{\mathcal{N}'} \tilde{\mathcal{O}}_{\mathcal{N}'}$, i.e. there exists a model agreement μ' for $\mathcal{N}'$ such that $\mathcal{I}_1 \in \mu'(\tilde{\mathcal{O}}_{\mathcal{N}'})$. We define families of interpretations $\{\mathcal{I}_m\}_{m \geqslant 1}$ and $\{\mathcal{I}'_m\}_{m \geqslant 1}$ such that for all $m \geqslant 1$, $\mathcal{I}_m \models \tilde{\mathcal{O}}_{\mathcal{N}'}$ and $\mathcal{I}'_m \models \mathcal{O}'$, and it holds $\mathcal{I}_m =_{\Sigma'} \mathcal{I}'_m$ and $\mathcal{I}'_m =_{\Sigma} \mathcal{I}_{m+1}$. The families of interpretations are defined by induction on m by showing that for any interpretation $\mathcal{I}_m$ such that $\mathcal{I}_m \models \tilde{\mathcal{O}}_{\mathcal{N}'} \cup \tilde{\mathcal{O}}$ there exist the corresponding interpretations $\mathcal{I}'_m$ and $\mathcal{I}_{m+1}$ such that both of them are also models of $\tilde{\mathcal{O}}$.

Given $\mathcal{I}_m$ as above, for $m \geqslant 1$, let $\mathcal{I}'_m$ be an interpretation, which agrees with $\mathcal{I}_m$ on Σ' and in which every A_i is interpreted as $(C'_i)^{\mathcal{I}_m}$, for $i = 1, \ldots, n$. Then $\mathcal{I}'_m \models \mathcal{O}'$ and since $\mathcal{I}_m \models B_i \equiv B'_i$, for $i = 1, \ldots, k$, we have $\mathcal{I}'_m \models \tilde{\mathcal{O}}'$. Then $\mathcal{I}'_m$ is a model of every concept inclusion obtained from an axiom of $\tilde{\mathcal{O}}'$ by substituting every occurrence of C'_i with A_i, for $i = 1, \ldots, n$, and therefore, from the definition of $\tilde{\mathcal{O}}$, we conclude that $\mathcal{I}'_m \models \tilde{\mathcal{O}}$. Now let $\mathcal{I}_{m+1}$ be an interpretation, which agrees with $\mathcal{I}'_m$ on Σ and in which every B'_i is interpreted as B_i, for $i = 1, \ldots, k$. Then $\mathcal{I}_{m+1}$ is a model of $\tilde{\mathcal{O}}_{\mathcal{N}'}$ and $\tilde{\mathcal{O}}$.

Since $\tilde{\mathcal{O}} \models \mathcal{O}$, we have $\mathcal{I}_m \models \mathcal{O}$, for all $m \geqslant 1$. As ontology $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible, there exists a model agreement μ for the network $\mathcal{N} \cup \{\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \texttt{sig}(\mathcal{O}), \mathcal{O}_{\mathcal{N}} \rangle\}$ such that $\mu(\tilde{\mathcal{O}}_{\mathcal{N}'}) = \{\mathcal{I}_m\}_{m \geqslant 1}$. Let us define a mapping μ' such that $\mu'(\mathcal{O}') = \{\mathcal{I}'_m\}_{m \geqslant 1}$ and the values of μ' and μ coincide on all other ontologies in $\mathcal{N}'$. Then μ' is the required model agreement for $\mathcal{N}'$. □

6 Hardness Results

We use reductions from the word problem for Turing machines (TMs) and alternating Turing machines (ATMs) to obtain most of the results in this section. We use the following conventions and notations related to these computation models. A Turing Machine (TM) is a tuple $M = \langle Q, \mathcal{A}, \delta \rangle$, where

Q is a set of states, with $\mathtt{q_h} \in Q$ being the accepting state, $\mathcal{A}$ is an alphabet, and $\delta : Q \times \mathcal{A} \mapsto Q \times \mathcal{A} \times \{-1, 1\}$ is a transition function. We assume w.l.o.g. that configuration of M is a word in the alphabet $Q \cup \mathcal{A}$ which contains exactly one state symbol $q \in Q$. An initial configuration is a word of the form $\mathtt{b} \dots \mathtt{b} \mathtt{q_0} \mathtt{b} \dots \mathtt{b}$, where $\mathtt{q_0} \in Q$ and $\mathtt{b} \in \mathcal{A}$ is the blank symbol. For a configuration $\mathfrak{c}$, the notion of successor configuration is defined by δ in a usual way and is denoted as $\delta(\mathfrak{c})$. A configuration $\mathfrak{c}$ is said to be accepting if there is a sequence of configurations $\mathfrak{c}_0, \dots, \mathfrak{c}_k$, $k \geqslant 0$, where $\mathfrak{c}_0 = \mathfrak{c}$, $\mathfrak{c}_k = v\mathtt{q_h}w$, and for all $0 \leqslant i < k$, $\mathfrak{c}_{i+1}$ is a successor of $\mathfrak{c}_i$. It is a well-known property of the transition functions of Turing machines that the symbol c'_i at position i of a configuration $\delta(\mathfrak{c})$ is uniquely determined by a 4-tuple of symbols $c_{i-2}, c_{i-1}, c_i, c_{i+1}$ at positions $i-2$, $i-1$, i, and $i+1$ of a configuration $\mathfrak{c}$. We assume that this correspondence is given by the (partial) function δ' and use the notation $c_{i-2}c_{i-1}c_ic_{i+1} \overset{\delta'}{\mapsto} c'_i$.

An Alternating Turing Machine (ATM) is a tuple $M = \langle Q, \mathcal{A}, \delta_1, \delta_2 \rangle$, where $Q = Q_\forall \cup Q_\exists \cup \{\mathtt{q_{rej}}\}$ is a set of states, with $\mathtt{q_{rej}} \in Q$ being the rejecting state, $\mathcal{A}$ is an alphabet containing the blank symbol $\mathtt{b} \in \mathcal{A}$, and for $\alpha = 1, 2$, $\delta_\alpha : Q \times \mathcal{A} \times Q \times \mathcal{A} \times \{-1, 1\}$ is a transition function. We assume w.l.o.g. that configuration of ATM M is a word in the alphabet $Q \cup \mathcal{A}$ which contains at most one symbol $q \in Q$. For a configuration $\mathfrak{c}$, the notion of successor configuration (wrt δ_α, $\alpha = 1, 2$) is defined in a usual way. A configuration $\mathfrak{c} = vqw$ is (inductively) defined as *rejecting* if either $q = \mathtt{q_{rej}}$, or $q \in Q_\forall$ and there is a successor configuration of $\mathfrak{c}$ which is rejecting, or $q \in Q_\exists$ and any successor configuration of $\mathfrak{c}$ is rejecting. A *rejecting run tree* of an ATM M for an initial configuration w is a tree in which the nodes are rejecting configurations of M, w is the root node, every child node is a successor configuration of its parent node, every leaf is a configuration with the state $\mathtt{q_{rej}}$, and if there is a node $\mathfrak{c} = uqw$, then the following holds: if $q \in Q_\forall$, then $\mathfrak{c}$ has a at least one child, and if $q \in Q_\exists$ then $\mathfrak{c}$ has two children. For $k \geqslant 0$, a configuration $\mathfrak{c}$ of M is said to be *k-rejecting* if M has a rejecting run tree of height k with the root $\mathfrak{c}$. The notions of *accepting run tree* (with every node not being a rejecting configuration) and *accepting configuration* are defined dually.

Similarly to ordinary TMs, we assume that the correspondence between a configuration $\mathfrak{c}$ and the successor configuration $\mathfrak{c}_\alpha$ of $\mathfrak{c}$ (wrt δ_α, $\alpha = 1, 2$) determined by 4-tuples of symbols is given by functions δ'_α, for $\alpha = 1, 2$. If every configuration of an ATM is a finite word of length n, then we assume w.l.o.g. that for $\alpha = 1, 2$ this correspondence is given as follows (for a word w of length n and $1 \leqslant i \leqslant n$, we denote by $w[i]$ the i-th symbol in w):

$$\mathfrak{c}[i-2]\mathfrak{c}[i-1]\mathfrak{c}[i]\mathfrak{c}[i+1] \overset{\delta'_\alpha}{\mapsto} \mathfrak{c}'[i], \quad \text{for } 1 \leqslant i \leqslant n-3$$

$$\mathtt{bb}\mathfrak{c}[1]\mathfrak{c}[2] \overset{\delta'_\alpha}{\mapsto} \mathfrak{c}'[1], \quad \text{for } \mathfrak{c}[1] \notin Q$$

$$\mathtt{b}\mathfrak{c}[1]\mathfrak{c}[2]\mathfrak{c}[3] \overset{\delta'_\alpha}{\mapsto} \mathfrak{c}'[2]$$

$$\mathfrak{c}[n-2]\mathfrak{c}[n-1]\mathfrak{c}[n]\mathtt{b} \overset{\delta'_\alpha}{\mapsto} \mathfrak{c}'[n]$$

Theorem 1. *Entailment in acyclic $\mathcal{EL}$-ontology networks is* ExpTime*-hard.*

Proof Sketch. We reduce the word problem for TMs making exponentially many steps to entailment in $\mathcal{EL}$-ontology networks. Let $M = \langle Q, \mathcal{A}, \delta \rangle$ be a TM and $n = \mathtt{1exp}(m)$ an exponential, for some $m \geqslant 0$. Consider an ontology $\mathcal{O}$ defined for M and n by the axioms (29)-(31) below.

The first axiom gives a r-chain containing $n+1$ segments of length $2n+3$, which are used to store fragments of consequent configurations of M:

$$A \sqsubseteq \exists r^{n \cdot (2n+3)}.(\mathtt{q_0} \sqcap \exists (r, \mathtt{b})^{2n+2}) \tag{29}$$

($A \notin Q \cup \mathcal{A}$).

We assume the following enumeration of segments in the r-chain:

$$\underbrace{\dots}_{s_n} \cdots \underbrace{\dots}_{s_1}\underbrace{\mathtt{q_0 b \dots b}}_{s_0}$$

i.e., s_0 represents a fragment of the initial configuration $\mathfrak{c}_0$ of M. For $0 \leqslant i < n$, every i-th and $(i+1)$-st segments in the r-chain are reserved for a pair of configurations $\mathfrak{c}_i, \mathfrak{c}_{i+1}$ such that $\mathfrak{c}_{i+1}$ is a successor of $\mathfrak{c}_i$.

The next family of axioms represents transitions of M and defines the 'content' of $(i+1)$-st segment based on the 'content' of i-th segment:

$$\exists r^{2n}(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W, \tag{30}$$

for all $X, Y, U, Z, W \in Q \cup \mathcal{A}$ such that $XYUZ \overset{\delta'}{\mapsto} W$.

Finally, the following axioms are used to initialise the halting marker H and propagate it to the 'left' of the r-chain:

$$\exists r.\mathtt{q_h} \sqsubseteq H, \;\; \exists r.H \sqsubseteq H \tag{31}$$

The definition of ontology $\mathcal{O}$ is complete. We claim that M accepts the empty word in n steps iff $\mathcal{O} \models A \sqsubseteq H$.

For the 'only if' direction we assume there is a sequence of configurations $\mathfrak{c}_0, \dots, \mathfrak{c}_n$ such that for all $0 \leqslant i < n$, $\mathfrak{c}_{i+1}$ is a successor configuration of $\mathfrak{c}_i$ and $\mathtt{q_h}$ is the state symbol in $\mathfrak{c}_n$. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and a domain element such that $a \in A^\mathcal{I}$. Then by axiom (29), there is an r-chain outgoing from x, which contains segments $s_0, \dots, s_n$ of length $2n+3$, where s_0 represents a fragment of $\mathfrak{c}_0$. It can be shown by induction that due to axioms (30), every segment s_i represents a fragment of $\mathfrak{c}_i$, for $1 \leqslant i \leqslant n$, and contains the state symbol from $\mathfrak{c}_i$. Then by axiom (31), it follows that $a \in H^\mathcal{I}$.

For the 'if' direction, one can provide a model $\mathcal{I}$ of $\mathcal{O}$ such that $A^\mathcal{I} = \{a\}$ is a singleton, $\mathtt{q_h}^\mathcal{I} = H^\mathcal{I} = \varnothing$, and $\mathcal{I}$ gives an r-chain outgoing from a, which contains $n+1$ segments representing fragments of consequent configurations of M, neither of which contains $\mathtt{q_h}$.

To complete the proof of the theorem we show that ontology $\mathcal{O}$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. Note that $\mathcal{O}$ contains axioms (29), (30) with concepts of size exponential in m. Consider axiom (29) and a concept inclusion φ of the form

$$A \sqsubseteq \exists r^{n \cdot (2n+3)}.B$$

where B is a concept name. Observe that it is equivalent to

$$A \sqsubseteq \underbrace{\exists r^p.\exists r^p}_{\text{2 times}}.\underbrace{\exists r^n.\exists r^n.\exists r^n}_{\text{3 times}}.B$$

where $p = \mathtt{1exp}(2m)$. Consider axiom ψ of the form $A \sqsubseteq B$. By iteratively applying Lemma 13 we obtain that $\psi[B \mapsto \exists r^p.\exists r^p.B]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. By repeating this argument we obtain the same for φ. Further, by Lemma 2, the axiom $\theta = \varphi[B \mapsto \mathtt{q_0} \sqcap B]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. Again, by iteratively applying Lemma 13 together with Lemma 2 we conclude that $\theta[B \mapsto \exists(r,\mathtt{b})^{2n+2}]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m and thus, so is axiom (29). The expressibility of axioms (30) is shown identically. The remaining axioms of ontology $\mathcal{O}$ are $\mathcal{EL}$-axioms whose size does not depend on m. By applying Lemma 1 we obtain that there exists an acyclic $\mathcal{EL}$-ontology network $\mathcal{N}$ of size polynomial in m and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq H$ iff M accepts the empty word in $\mathtt{1exp}(m)$ steps. □

Theorem 2. *Entailment in cyclic $\mathcal{EL}$-ontology networks is RE-hard.*

Proof Sketch. For a TM $M = \langle Q, \mathcal{A}, \delta\rangle$, we define an infinite ontology $\mathcal{O}$, which contains variants of axioms (29-30) from Theorem 1 and additional axioms for a correct implementation of transitions of M.

Axioms (32) give an infinite family of r-chains, each having a 'prefix' of length $k+1$, for $k \geqslant 0$ (reserved for fragments of consequent configurations of M), and a 'postfix' containing a chain of length $2l+3$, for $l \geqslant 0$, which represents a fragment of the initial configuration $\mathfrak{c}_0$:

$$A \sqsubseteq \exists r^k.(\exists v^l.L \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists(r,\mathtt{b})^{2l+2})) \tag{32}$$

for $A, L, \varepsilon \notin Q \cup \mathcal{A}$ and all $k, l \geqslant 0$.

Propagated to the 'left' by the next family of axioms, concept $\exists v^k.L$, $k \geqslant 0$, indicates the length of the 'postfix' for $\mathfrak{c}_0$ on every r-chain given by axioms (32):

$$\exists r.\exists v^k.L \sqsubseteq \exists v^k.L, \quad k \geqslant 0 \tag{33}$$

The concept ε is used to separate fragments of consequent configurations of M and is therefore propagated as follows:

$$\exists v^k.L \sqcap \exists r^{2k+4}.\varepsilon \sqsubseteq \varepsilon, \quad k \geqslant 0 \tag{34}$$

The next families of axioms, with $X, Y, U, Z, W \in Q \cup \mathcal{A}$, implement transitions of M.

$$\exists v^k.L \sqcap \sqcap \exists r^{2k+1}.(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W \tag{35}$$

for $XYUZ \overset{\delta'}{\mapsto} W$ and all $k \geqslant 0$.

Concept $\exists v^k.L$ guarantees that transitions have effect only along r-chains which represent a fragment of $\mathfrak{c}_0$ of length $2k+3$. Since $\varepsilon \notin Q \cup \mathcal{A}$, the transitions involving ε are implemented separately by the following families of axioms:

$$\exists v^k.L \sqcap \sqcap \exists r^{2k}.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W \tag{36}$$

for $\mathtt{b}YUZ \overset{\delta'}{\mapsto} W$ and all $k \geqslant 0$;

$$\exists v^k.L \sqcap \sqcap \exists r^{2k}.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W \tag{37}$$

for $\mathtt{bb}UZ \overset{\delta'}{\mapsto} W$ and all $k \geqslant 0$;

$$\exists v^k.L \sqcap \sqcap \exists r^{2k}.(X \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.Z)))) \sqsubseteq W \tag{38}$$

for $X\mathtt{bb}Z \overset{\delta'}{\mapsto} W$ and all $k \geqslant 0$.

The last axiom of $\mathcal{O}$ is used to initialize the halting marker H and propagate it to the 'left' of a r-chain:

$$\mathtt{q_h} \sqsubseteq H, \quad \exists r.H \sqsubseteq H \tag{39}$$

The definition of ontology $\mathcal{O}$ is complete.

The more involved implementation of transitions (in comparison to Theorem 1) allows to prevent defect situations, when there are two consequent segments s_i,s_{i+1} of an r-chain, which represent fragments of configurations $\mathfrak{c}_i$,$\mathfrak{c}_{i+1}$ of M, respectively, but $\mathfrak{c}_{i+1}$ is not a successor of $\mathfrak{c}_i$. In Theorem 1, the prefix of length $n \cdot (2n+3)$ given by axiom (29) guarantees a correct implementation of up to n transitions of the TM. The situation is different in the infinite case, since the prefix reserved for fragments of consequent configurations of M can be of any length, due to axioms (32).

We prove that M halts iff $\mathcal{O} \models A \sqsubseteq H$. Suppose that $\mathfrak{c}_0$ is an accepting configuration and M halts in n steps; w.l.o.g. we assume that $n > 1$. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and $a \in A^\mathcal{I}$ a domain element. Due to axioms (32), $\mathcal{I}$ is a model of the concept inclusion:

$$A \sqsubseteq \exists r^{n\cdot(2n+4)}.(\exists v^n.L \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists(r,\mathtt{b})^{2n+2}))$$

and thus, $\mathcal{I}$ gives a r-chain containing $n+1$ segments of length $2n+3$ separated by ε. By using arguments from the proof of Theorem 1, it can be shown that due to axioms (33) - (38), these segments represent fragments of consequent configurations of M, starting with $\mathfrak{c}_0$, and there is an element b in the r-chain such that $b \in \mathtt{q_h}^\mathcal{I}$. Then by axiom (39), it holds $a \in H^\mathcal{I}$.

For the 'if' direction, one can show that if M does not halt, then there exists a model $\mathcal{I}$ of $\mathcal{O}$ such that $\mathtt{q_h}^\mathcal{I} = H^\mathcal{I} = \varnothing$, $A^\mathcal{I} = \{a\}$ is a singleton and there are infinitely many disjoint r-chains $\{R_{m,n}\}_{m,n \geqslant 1}$ outgoing from a, such that every $R_{m,n}$ represents a fragment of $\mathfrak{c}_0$ of length n and has a prefix of length $m+1$ representing fragments of consequent configurations of M, each having length $\leqslant 2n+3$.

To complete the proof of the theorem we show that ontology $\mathcal{O}$ is expressible by a cyclic $\mathcal{EL}$-ontology network. Let us demonstrate that so is the family of axioms (32). Let $\varphi = A \sqsubseteq B$ be a concept inclusion and $B, B1, B_2$ concept names. By Lemma 14, ontology $\mathcal{O}_1 = \{\varphi[B \mapsto \exists r^k.B] \mid k \geqslant 0\}$ is expressible by a cyclic $\mathcal{EL}$-ontology network. Then by Lemma 2, ontology $\mathcal{O}_2 = \mathcal{O}_1[B \mapsto B_1 \sqcap \varepsilon \sqcap \exists r.(\mathsf{q}_0 \sqcap B_2)]$ is expressible by a cyclic $\mathcal{EL}$-ontology network. By applying Lemma 14 again, we conclude that so is ontology $\mathcal{O}_3 = \bigcup_{l \geqslant 0} \mathcal{O}_2[B_1 \mapsto \exists v^l.B_1,\ B_2 \mapsto \exists(r,\mathsf{b})^{2l}.B_2]$, i.e., the ontology given by axioms

$$A \sqsubseteq \exists r^k.(\exists v^l.B_1 \sqcap \varepsilon \sqcap \exists r.(\mathsf{q}_0 \sqcap \exists(r,\mathsf{b})^{2l}.B_2))$$

for $k, l \geqslant 0$. Further, by Lemma 2, we obtain that $\mathcal{O}_2[B_1 \mapsto L,\ B_2 \mapsto \exists(r,\mathsf{b})^2]$ is expressible by a cyclic $\mathcal{EL}$-ontology network and hence, so is the family of axioms (32). A similar argument shows the expressibility of ontologies given by axioms (33)-(38). The remaining subset of axioms (39) of $\mathcal{O}$ is finite. By Lemma 1, there exists a cyclic $\mathcal{EL}$-ontology network $\mathcal{N}$ and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq H$ iff M halts. $\square$

Theorem 3. *Entailment in $\mathcal{ALC}$-ontology networks is 2ExpTime-hard.*

Proof Sketch. The result is based on the construction from the proof of Theorem 7 in [Kazakov, 2008], where it is shown that the word problem for $\mathtt{1exp}(n)$-space bounded ATMs, for $n \geqslant 0$, reduces to satisfiability of $\mathcal{R}$-ontologies. We demonstrate that under a minor modification the construction used in that theorem shows that there is a $\mathcal{ALC}$-ontology $\mathcal{O}$ containing nested concepts of exponential size and a concept name A such that $\mathcal{O} \not\models A \sqsubseteq \bot$ iff a given $\mathtt{1exp}(n)$-space bounded ATM accepts the empty word. The ontology contains axioms with concepts of the form $\exists(r,C)^{\mathtt{1exp}(n)}.D$ and $\forall r^{\mathtt{1exp}(n)}.D$. Using axioms of the form $Z \sqsubseteq \exists(r,C)^{\mathtt{1exp}(n)}.D$ it is possible to encode consequent exponentially long r-chains for storing consequent configurations of ATM. With axioms of the form $Z \sqsubseteq \forall r^{\mathtt{1exp}(n)}.D$ it is possible to encode transitions between configurations by defining correspondence of interpretations of concept names (encoding the alphabet of ATM) on two consequent $\mathtt{1exp}(n)$-long r-chains. The rest of the concept inclusions in $\mathcal{O}$ are $\mathcal{ALC}$-axioms, whose size does not depend on n and which are used to represent the initial configuration, existential/universal types configurations, and describe additional conditions for implementation of transitions. By using Lemmas 4, 12, we demonstrate that every axiom of $\mathcal{O}$ containing concepts of size exponential in n is expressible by an acyclic $\mathcal{ALC}$-ontology network $\mathcal{N}$ of size polynomial in n. Then by applying Lemma 1 we obtain that there exists an acyclic $\mathcal{ALC}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff M accepts the empty word. Since AExpSpace = 2ExpTime, we obtain the required statement. $\square$

Theorem 4. *Entailment in $\mathcal{R}$-ontology networks is 3ExpTime-hard.*

Proof Sketch. The proof is by reduction of the word problem for $\mathtt{2exp}(n)$-space bounded ATMs to entailment in $\mathcal{R}$-ontology networks. Given such TM M and a number $n \geqslant 0$, we consider ontology $\mathcal{O}$ from the proof of Theorem 3 for M and let $\mathcal{O}'$ be the ontology obtained from $\mathcal{O}$ by replacing every nested concept of the form $\exists(r,C)^{\mathtt{1exp}(n)}.D$ and $\forall r^{\mathtt{1exp}(n)}.D$ with $\exists(r,C)^{\mathtt{2exp}(n)}.D$ and $\forall r^{\mathtt{2exp}(n)}.D$, respectively. Then a repetition of the proof of Theorem 3 gives that $\mathcal{O}' \models A \sqsubseteq \bot$ iff M accepts the empty word. By applying Lemmas 10, 12 we show that every axiom of $\mathcal{O}'$ containing concepts of size double exponential in n is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n. The remaining axioms of $\mathcal{O}'$ are $\mathcal{ALC}$ axioms, whose size does not depend on n. Then by applying Lemma 1 we obtain that there exists an acyclic $\mathcal{R}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}'$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff M accepts the empty word. Since A2ExpSpace = 3ExpTime, we obtain the required statement. $\square$

Theorem 5. *Entailment in $\mathcal{ALCHOIF}$-ontology networks is coN2ExpTime-hard.*

Proof Sketch. The result is based on the construction from the proof of Theorem 5 in [Kazakov, 2008], where it is shown that the N2ExpTime-hard problem of existence of a domino tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$, $n \geqslant 0$, reduces to satisfiability of $\mathcal{ROIF}$-ontologies. We demonstrate that under a minor modification the construction used in that theorem shows that there exists a $\mathcal{ALCHOIF}$-ontology $\mathcal{O}$ containing concepts of an exponential size and a concept name A such that $\mathcal{O} \not\models A \sqsubseteq \bot$ iff a given domino system admits a tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$, for $n \geqslant 0$. Ontology $\mathcal{O}$ contains axioms with concepts of the form $\exists r^{\mathtt{1exp}(n)}.C$ and $\forall r^{\mathtt{1exp}(n)}.C$. Axioms of the form $Z \sqsubseteq \exists r^{\mathtt{1exp}(n)}.C$ allow one to encode $\mathtt{1exp}(n)$-long r-chains. Using a variant of the binary counter technique together with axioms of the form $Z \sqsubseteq \forall r^{\mathtt{1exp}(n)}.C$ and role hierarchies allows one to encode $\mathtt{2exp}(n)$-many consequent r-chains of this kind, thus obtaining sequences of $\mathtt{2exp}(n)$-many end points of r-chains. With nominals and inverse functional roles it is possible to enforce coupling of these sequences to obtain a grid of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$. Finally, $\mathcal{ALC}$ axioms with concepts of the form $\forall r^{\mathtt{1exp}(n)}.C$ allow one to represent the initial and matching conditions of the domino tiling problem. By using the same arguments as in the proof of Theorem 3 we show that there is a $\mathcal{ALCHOIF}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff the domino system does not admit a tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$. $\square$

Theorem 6. *Entailment in $\mathcal{ROIF}$-ontology networks is coN3ExpTime-hard.*

Proof Sketch. The theorem is proved by a reducing the N3ExpTime-hard problem of domino tiling of size $\mathtt{3exp}(n) \times \mathtt{3exp}(n)$ to entailment in $\mathcal{ROIF}$-ontology networks. Given an instance of this problem, we consider ontology $\mathcal{O}$ defined in the proof of Theorem 5 and let $\mathcal{O}'$ be the ontology obtained from $\mathcal{O}$ by replacing every nested concept $\exists r^{\mathtt{1exp}(n)}.C$ and $\forall r^{\mathtt{1exp}(n)}.C$ with $\exists r^{\mathtt{2exp}(n)}.C$ and $\forall r^{\mathtt{2exp}(n)}.C$, respectively. A repetition of the proof of Theorem 5 shows that $\mathcal{O} \not\models A \sqsubseteq \bot$ iff a given domino system admits a tiling of size $3\texttt{exp}(n) \times 3\texttt{exp}(n)$. By applying Lemmas 1, 10, 12 we obtain that there exists a $\mathcal{ROIF}$-ontology network $\mathcal{N}$ of a polynomial size and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff the given domino system does not admit a tiling of size $3\texttt{exp}(n) \times 3\texttt{exp}(n)$. $\Box$

Theorem 7. *Entailment in $\mathcal{H}$-Networks is ExpTime-hard.*

Proof Sketch. We show that the word problem for ATMs working with words of a polynomial length n reduces to entailment in cyclic $\mathcal{H}$-ontology networks. Then, since APSpace = ExpTime, the claim follows.

Let $M = \langle Q, \mathcal{A}, \delta_1, \delta_2\rangle$ be an ATM. We call the word of the form $\texttt{bq}_0\texttt{b}\ldots\texttt{b}$ *initial configuration* of M. Consider a signature σ consisting of concept names B_{ai}, for $a \in Q \cup \mathcal{A}$ and $1 \leqslant i \leqslant n$ (with the informal meaning that the i-th symbol in a configuration of M is a). Let σ^1, and σ^2 be 'copies' of signature σ consisting of the above mentioned concept names with the superscripts ^{1} and ^{2}, respectively.

For $\alpha = 1, 2$, let $\mathcal{O}^\alpha$ be an ontology consisting of the axioms below. The family of axioms (40)-(43) implements transitions of M (while respecting the end positions of configurations):

$$B^\alpha_{Xi-2} \sqcap B^\alpha_{Yi-1} \sqcap B^\alpha_{Ui} \sqcap B^\alpha_{Vi+1} \sqsubseteq B_{Wi} \qquad (40)$$

for $1 \leqslant i \leqslant n-3$ and all $X, Y, U, V, W \in Q \cup \mathcal{A}$ such that $XYUV \overset{\delta_\alpha}{\mapsto} W$;

$$B^\alpha_{U1} \sqcap B^\alpha_{V2} \sqsubseteq B_{W1} \qquad (41)$$

for all $U, V, W \in Q \cup \mathcal{A}$ such that $\texttt{bb}UV \overset{\delta_\alpha}{\mapsto} W$;

$$B^\alpha_{Y1} \sqcap B^\alpha_{U2} \sqcap B^\alpha_{V3} \sqsubseteq B_{W2} \qquad (42)$$

for all $Y, U, V, W \in Q \cup \mathcal{A}$ such that $\texttt{b}YUV \overset{\delta_\alpha}{\mapsto} W$;

$$B^\alpha_{Xn-2} \sqcap B^\alpha_{Yn-1} \sqcap B^\alpha_{Un} \sqsubseteq B_{Wn} \qquad (43)$$

for all $X, Y, U, W \in Q \cup \mathcal{A}$ such that $XYU\texttt{b} \overset{\delta_\alpha}{\mapsto} W$.

For $1 \leqslant i \leqslant n$, the next axioms initialize 'local' marker $\bar{H}^\alpha$ and 'global' marker $\bar{H}$ for a rejecting successor configuration wrt δ_α:

$$B_{\texttt{q}_{\texttt{rej}}i} \sqsubseteq \bar{H}, \quad \bar{H} \sqsubseteq \bar{H}^\alpha \qquad (44)$$

Let $\mathcal{O}$ be an ontology consisting of the following axioms:

$$\bar{H}^1 \sqcap B_{q_\forall i} \sqsubseteq \bar{H}, \;\; \bar{H}^2 \sqcap B_{q_\forall i} \sqsubseteq \bar{H} \qquad (45)$$
$$\bar{H}^1 \sqcap \bar{H}^2 \sqcap B_{q_\exists i} \sqsubseteq \bar{H}$$

for $1 \leqslant i \leqslant m$, $\texttt{q}_\exists \in Q_\exists$, and $\texttt{q}_\forall \in Q_\forall$ (i.e., these axioms implement the definition of accepting configuration depending on whether the state is existential or universal);

$$A \sqsubseteq \textstyle\bigsqcap_{1 \leqslant i \leqslant n+2} B_{\texttt{b}i} \sqcap B_{q_0 n+3} \sqcap \bigsqcap_{n+4 \leqslant i \leqslant m} B_{\texttt{b}i} \qquad (46)$$

representing the initial configuration $\mathfrak{c}_{\texttt{init}}$ of M;

$$B_{ai} \sqsubseteq B^\alpha_{ai} \qquad (47)$$

for $\alpha = 1, 2$, $1 \leqslant i \leqslant n$, and all $a \in Q \cup \mathcal{A}$ (which enforce 'copying' a configuration 'description' in signature σ into signatures σ^1, σ^2).

Consider ontology network $\mathcal{N}$ consisting of the import relations $\langle \mathcal{O}, \Sigma^\alpha, \mathcal{O}^\alpha\rangle$ and $\langle \mathcal{O}^\alpha, \Sigma, \mathcal{O}\rangle$, where $\Sigma^\alpha = \{\bar{H}^\alpha\} \cup \sigma^\alpha$, $\Sigma = \{\bar{H}\} \cup \sigma$, and $\alpha = 1, 2$. Informally, ontologies $\mathcal{O}^\alpha$ describe transitions between configurations, while $\mathcal{O}$ serves for 'copying' configuration descriptions into signatures σ^1, σ^2 and 'feeding' them back into $\mathcal{O}^\alpha$. It follows that models $\mathcal{I} \models_\mathcal{N} \mathcal{O}$ represent consequent configurations of M and thus, network $\mathcal{N}$ implements a run tree of ATM. Intuitively, a point x in the domain of a model $\mathcal{I} \models_\mathcal{N} \mathcal{O}$ represents a configuration $\mathfrak{c}$, if x belongs to the interpretation of σ-concept names, corresponding to the symbols in $\mathfrak{c}$.

We demonstrate that M does not accept the empty word iff $\mathcal{O} \models_\mathcal{N} A \sqsubseteq \bar{H}$. The 'only if' direction is proved by induction by showing that in any model $\mathcal{I} \models_\mathcal{N} \mathcal{O}$, whenever a domain element x represents a rejecting configuration $\mathfrak{c}$ of M, it holds $x \in \bar{H}^\mathcal{I}$. Then it follows by axiom (46) that $x \in \bar{H}^\mathcal{I}$, whenever $x \in A^\mathcal{I}$. The 'if' direction is proved by contraposition by defining a model agreement μ for $\mathcal{N}$ and a singleton interpretation $\mathcal{I} \in \mu(\mathcal{O})$ such that $\mathcal{I} \not\models A \sqsubseteq \bar{H}$. For every ontology, μ gives a family of singleton interpretations such that each of them represents a configuration of M and agreed interpretations correspond to consequent configurations. $\Box$

Theorem 8. *Entailment in acyclic $\mathcal{H}$-Networks is PSpace-hard.*

Proof Sketch. We show that the word problem for ATMs making polynomially many steps reduces to entailment in acyclic $\mathcal{H}$-ontology networks. Then, since AP = PSpace, the claim follows.

Let $M = \langle Q, \mathcal{A}, \delta_1, \delta_2\rangle$ be an ATM. We use the definition of the network $\mathcal{N}$ from the proof sketch to Theorem 7 and define by induction an acyclic $\mathcal{H}$-ontology network $\mathcal{N}_n$, which can be viewed informally as a finite 'unfolding' of $\mathcal{N}$.

For $n = 1$, let $\mathcal{N}_1$ be a network consisting of import relations $\langle \mathcal{O}_1, \Sigma^\alpha, \mathcal{O}^\alpha_1\rangle$, for $\alpha = 1, 2$, where $\mathcal{O}_1$ is equivalent to ontology $\mathcal{O}$ (from the definition of network $\mathcal{N}$) and $\mathcal{O}^\alpha_1$ is equivalent to $\mathcal{O}^\alpha$. If $\mathcal{N}_{n-1}$ is a network already given for $n \geqslant 2$, then we define $\mathcal{N}_n$ as the union of $\mathcal{N}_{n-1}$ with the set consisting of import relations $\langle \mathcal{O}_n, \Sigma^\alpha, \mathcal{O}^\alpha_n\rangle$, $\langle \mathcal{O}^\alpha_n, \Sigma, \mathcal{O}_{n-1}\rangle$, for $\alpha = 1, 2$, where $\mathcal{O}_n$, $\mathcal{O}^\alpha_n$ are ontologies not present in $\mathcal{N}_{n-1}$ and $\mathcal{O}_n$ is equivalent to $\mathcal{O}$ and $\mathcal{O}^\alpha_n$ is equivalent to $\mathcal{O}^\alpha$. By using the arguments from the proof of Theorem 7 we show that for any $n \geqslant 1$, it holds $\mathcal{O}_n \models_{\mathcal{N}_n} A \sqsubseteq \bar{H}$ iff M does not accept the empty word in n steps. $\Box$

7 Reduction to Classical Entailment

As a tool for proving upper complexity bounds, we demonstrate that entailment in a network $\mathcal{N}$ can be reduced to entailment from (a possibly infinite) union of 'copies' of ontologies appearing in $\mathcal{N}$.

Let $\mathcal{N}$ be an ontology network. We denote $\texttt{sig}(\mathcal{N}) = \bigcup_{\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2\rangle \in \mathcal{N}} (\texttt{sig}(\mathcal{O}_1) \cup \Sigma \cup \texttt{sig}(\mathcal{O}_2))$. An *import path* in $\mathcal{N}$ is a sequence $p = \{\mathcal{O}_0, \Sigma_1, \mathcal{O}_1, \ldots, \mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n\}$, $n \geq 0$, such that $\langle \mathcal{O}_{i-1}, \Sigma_i, \mathcal{O}_i\rangle \in \mathcal{N}$ for each i with $(1 \leq i \leq n)$. We denote by $\texttt{len}(p) = n$, $\texttt{first}(p) = \mathcal{O}_0$ and $\texttt{last}(p) = \mathcal{O}_n$ the *length* of p, the *first* and, respectively, the *last* ontologies on the path p. By $\texttt{paths}(\mathcal{N})$ we define the set of all paths in $\mathcal{N}$, and by $\texttt{paths}(\mathcal{N}, \mathcal{O}) = \{p \in$

$\text{paths}(\mathcal{N}) \mid \text{first}(p) = \mathcal{O}\}$ the subset of paths that originate in $\mathcal{O}$. We say that $\mathcal{O}'$ is *reachable from* $\mathcal{O}$ *in* $\mathcal{N}$ if there exists a path $p \in \text{paths}(\mathcal{N}, \mathcal{O})$ such that $\text{last}(p) = \mathcal{O}'$. The *import closure* of an ontology $\mathcal{O}$ in $\mathcal{N}$ is defined by $\bar{\mathcal{O}} = \cup_{p \in \text{paths}(\mathcal{N},\mathcal{O})} \text{last}(p)$. Note that by definition it holds $\{\mathcal{O}\} \in \text{paths}(\mathcal{N}, \mathcal{O})$ and thus, $\mathcal{O} \subseteq \bar{\mathcal{O}}$.

Lemma 15. *If* $\mathcal{I} \models \bar{\mathcal{O}}$ *then* $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$.

Proof. Consider a mapping μ defined for ontologies $\mathcal{O}'$ in $\mathcal{N}$ by setting $\mu(\mathcal{O}') = \{\mathcal{I}\}$ if $\mathcal{O}'$ is reachable from $\mathcal{O}$ and $\mu(\mathcal{O}') = \emptyset$ otherwise. Clearly, μ is a model agreement for $\mathcal{N}$. Since $\mathcal{I} \in \mu(\mathcal{O})$, we have $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$. □

For every symbol $X \in \text{sig}(\mathcal{N})$ and every import path p in $\mathcal{N}$, take a distinct symbol X_p of the same type (concept name, role name, or individual) not occurring in $\text{sig}(\mathcal{N})$. For each import path p in $\mathcal{N}$, define a *renaming* θ_p of symbols in $\text{sig}(\mathcal{N})$ inductively as follows. If $\text{len}(p) = 0$, we set $\theta_p(X) = X$ for every $X \in \text{sig}(\mathcal{N})$. Otherwise, $p = p' \cup \{\mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n\}$ for some path p' and we define $\theta_p(X) = \theta_{p'}(X)$ if $X \in \Sigma_n$ and $\theta_p(X) = X_p$ otherwise. A *renamed import closure* of an ontology $\mathcal{O}$ in $\mathcal{N}$ is defined by $\tilde{\mathcal{O}} = \bigcup_{p \in \text{paths}(\mathcal{N},\mathcal{O})} \theta_p(\text{last}(p))$.

Lemma 16. *If* $\mathcal{I} \models \tilde{\mathcal{O}}$ *then* $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$.

Proof. The proof is identical to the proof of Lemma 15. □

Lemma 17. *For every* $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$ *there exists* $\mathcal{J} \models \tilde{\mathcal{O}}$ *such that* $\mathcal{J} =_{\text{sig}(\mathcal{N})} \mathcal{I}$.

Proof. Assume that $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$. Then there exists a model agreement μ for $\mathcal{N}$ such that $\mathcal{I} \in \mu(\mathcal{O})$. We define $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ by setting $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$, and $X^{\mathcal{J}} = X^{\mathcal{I}}$ for all symbols X except for the symbols X_p, with $p \in \text{paths}(\mathcal{N}, \mathcal{O})$ and $X \in \text{sig}(\mathcal{N})$. For those symbols, we set $(X_p)^{\mathcal{J}} = X^{\mathcal{I}_p}$ where $\mathcal{I}_p \in \mu(\text{last}(p))$ is defined by induction on $\text{len}(p)$ as follows. If $\text{len}(p) = 0$, we set $\mathcal{I}_p = \mathcal{I} \in \mu(\mathcal{O}) = \mu(\text{last}(p))$. Otherwise, $p = p' \cup \{\mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n\}$ for some $\langle \mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n \rangle \in \mathcal{N}$, and $\mathcal{I}_{p'} \in \mu(\mathcal{O}_{n-1})$ is already defined. Then pick any $\mathcal{I}_p \in \mu(\mathcal{O}_n)$ such that $\mathcal{I}_p =_{\Sigma_n} \mathcal{I}_{p'}$. Such $\mathcal{I}_p$ always exists since μ is a model agreement. This completes the definition of $\mathcal{J}$. Obviously, $\mathcal{J} =_{\text{sig}(\mathcal{N})} \mathcal{I}$.

To prove that $\mathcal{J} \models \tilde{\mathcal{O}}$, we first show by induction on $\text{len}(p)$ that for every $X \in \text{sig}(\mathcal{N})$ we have $(\theta_p(X))^{\mathcal{J}} = X^{\mathcal{I}_p}$. Indeed, if $\text{len}(p) = 0$ then $(\theta_p(X))^{\mathcal{J}} = X^{\mathcal{J}} = X^{\mathcal{I}} = X^{\mathcal{I}_p}$. If $p = p' \cup \{\mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n\}$ for some $\langle \mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n \rangle \in \mathcal{N}$, then if $X \in \Sigma_n$, we have $(\theta_p(X))^{\mathcal{J}} = (\theta_{p'}(X))^{\mathcal{J}} = X^{\mathcal{I}_{p'}} = X^{\mathcal{I}_p}$ since $\mathcal{I}_p =_{\Sigma_n} \mathcal{I}_{p'}$. If $X \notin \Sigma_n$ then $(\theta_p(X))^{\mathcal{J}} = (X_p)^{\mathcal{J}} = X^{\mathcal{I}_p}$.

Now, since for every path $p \in \text{paths}(\mathcal{N}, \mathcal{O})$, we have $\mathcal{I}_p \in \mu(\text{last}(p))$ hence, in particular, $\mathcal{I}_p \models \text{last}(p)$, we have $\mathcal{J} \models \theta_p(\text{last}(p))$ by the property above. Hence $\mathcal{J} \models \tilde{\mathcal{O}}$. □

Theorem 9. *Let* $\mathcal{N}$ *be an ontology network,* $\mathcal{O}$ *an ontology in* $\mathcal{N}$*, and* α *an axiom such that* $\text{sig}(\alpha) \subseteq \text{sig}(\mathcal{N})$*. Then* $\mathcal{O} \models_{\mathcal{N}} \alpha$ *iff* $\tilde{\mathcal{O}} \models \alpha$.

Proof. Suppose that $\mathcal{O} \models_{\mathcal{N}} \alpha$. In order to prove that $\tilde{\mathcal{O}} \models \alpha$, take any model $\mathcal{I} \models \tilde{\mathcal{O}}$. We need to show that $\mathcal{I} \models \alpha$. Since $\mathcal{I} \models \tilde{\mathcal{O}}$, by Lemma 16, we have $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$. Since $\mathcal{O} \models_{\mathcal{N}} \alpha$, we have $\mathcal{I} \models \alpha$, as required.

Conversely, suppose that $\tilde{\mathcal{O}} \models \alpha$. In order to show that $\mathcal{O} \models_{\mathcal{N}} \alpha$, take any $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$. We need to show that $\mathcal{I} \models \alpha$. Since $\mathcal{I} \models_{\mathcal{N}} \mathcal{O}$, by Lemma 17, there exists $\mathcal{J} \models \tilde{\mathcal{O}}$ such that $\mathcal{J} =_{\text{sig}(\mathcal{N})} \mathcal{I}$. Since $\tilde{\mathcal{O}} \models \alpha$, we have $\mathcal{J} \models \alpha$. Since $\text{sig}(\alpha) \subseteq \text{sig}(\mathcal{N})$, we have $\mathcal{I} \models \alpha$, as required. □

8 Membership Results

Theorem 9 provides a method for reducing the entailment problem in ontology networks to entailment from ontologies. Note that, in general, the renamed closure $\tilde{\mathcal{O}}$ of an ontology $\mathcal{O}$ in a (cyclic) network $\mathcal{N}$ can be infinite (even if $\mathcal{N}$ and all ontologies in $\mathcal{N}$ are finite). There are, however, special cases when $\tilde{\mathcal{O}}$ is finite. For example, if all import signatures in $\mathcal{N}$ include all symbols in $\text{sig}(\mathcal{N})$, then it is easy to see that $\tilde{\mathcal{O}} = \bar{\mathcal{O}}$. $\tilde{\mathcal{O}}$ is also finite if $\text{paths}(\mathcal{N}, \mathcal{O})$ is finite, e.g., if $\mathcal{N}$ is acyclic. In this case, the size of $\tilde{\mathcal{O}}$ is at most exponential in $\mathcal{O}$. If there is at most one import path between every pair of ontologies (i.e., if $\mathcal{N}$ is tree-shaped) then the size of $\tilde{\mathcal{O}}$ is the same as the size of $\mathcal{N}$. This immediately gives the upper complexity bounds on deciding entailment in acyclic networks.

Theorem 10. *Let* $\mathcal{L}$ *be a DL with the complexity of entailment in* **[co][N]TIME**$(f(n))$ *(***[co]*** and* **[N]** *denote possible co- and N-prefix, respectively). Let* $\mathcal{N}$ *be an acyclic ontology network and* $\mathcal{O}$ *an ontology in* $\mathcal{N}$ *such that* $\tilde{\mathcal{O}}$ *is a* $\mathcal{L}$*-ontology. Then for* $\mathcal{L}$*-axioms* α*, the entailment* $\mathcal{O} \models_{\mathcal{N}} \alpha$ *is decidable in* **[co][N]TIME**$(f(2^n))$*. If* $\mathcal{N}$ *is tree-shaped then deciding* $\mathcal{O} \models_{\mathcal{N}} \alpha$ *has the same complexity as entailment in* $\mathcal{L}$.

Note that if $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are some ontologies in a DL $\mathcal{L}$, then in general, their union is not necessary a $\mathcal{L}$-ontology. For instance, this is the case for logics containing DL $\mathcal{R}$, which restricts role inclusion axioms to regular ones. The regularity property can be easily lost when taking the union of ontologies and thus, reasoning over the union of ontologies may be harder than reasoning in the underlying DL. Hence, the requirement in Theorem 10 that $\tilde{\mathcal{O}}$ must be a $\mathcal{L}$-ontology.

In the next theorem, we show that for arbitrary networks, checking entailment is, in general, semi-decidable, which is a consequence of the Compactness Theorem for First-Order Logic (since all standard DLs can be translated to FOL).

Theorem 11. *Let* $\mathcal{L}$ *be a DL, which can be translated to FOL,* $\mathcal{N}$ *an ontology network, and* $\mathcal{O}$ *an ontology in* $\mathcal{N}$ *such that* $\tilde{\mathcal{O}}$ *is a* $\mathcal{L}$*-ontology. Then for* $\mathcal{L}$*-axioms* α*, the entailment* $\mathcal{O} \models_{\mathcal{N}} \alpha$ *is semi-decidable.*

Proof. By Theorem 9, $\mathcal{O} \models_{\mathcal{N}} \alpha$ iff $\tilde{\mathcal{O}} \models \alpha$. By the compactness theorem for first-order logic, if $\tilde{\mathcal{O}} \models \alpha$ then there exists a finite subset $\mathcal{O}' \subseteq \tilde{\mathcal{O}}$ such that $\mathcal{O}' \models \alpha$. Hence, $\tilde{\mathcal{O}} \models \alpha$ can be checked, e.g, by enumerating all finite subsets $\tilde{\mathcal{O}}_n = \bigcup_{p \in \text{paths}(\mathcal{N},\mathcal{O},n)} \theta_p(\text{last}(p)) \subseteq \tilde{\mathcal{O}}$, $n \geq 0$, where $\text{paths}(\mathcal{N}, \mathcal{O}, n) = \{p \in \text{paths}(\mathcal{N}, \mathcal{O}) \mid \text{len}(p) \leq n\}$ and running the (semi-decidable) test $\tilde{\mathcal{O}}_n \models \alpha$ with the timeout n. If $\tilde{\mathcal{O}} \models \alpha$ then, eventually, one of these tests succeeds. $\square$

Restricting the shape of the network is one possibility of establishing decidability results for entailment in ontology networks. Another possibility is to restrict the language. It turns out, for ontology networks expressed in the role-free DL $\mathcal{P}$, the entailment problem becomes decidable, even in the presence of cycles. Intuitively, this is because the entailment in $\mathcal{P}$ can be characterized by a bounded number of models.

Definition 3. *We say that an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a* singleton *if $\|\Delta^{\mathcal{I}}\| = 1$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a DL interpretation and $d \in \Delta^{\mathcal{I}}$. The* singleton projection *of $\mathcal{I}$ to d is the interpretation $\mathcal{J} = (\{d\}, \cdot^{\mathcal{J}})$ such that $A^{\mathcal{J}} = A^{\mathcal{I}} \cap \{d\}$ for each $A \in \mathsf{N_C}$, $R^{\mathcal{J}} = \emptyset$ for each $R \in \mathsf{N_R}$, and $a^{\mathcal{J}} = d$ for each $a \in \mathsf{N_i}$.*

Lemma 18. *Let C be a $\mathcal{P}$-concept, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation, and $\mathcal{J} = (\{d\}, \cdot^{\mathcal{J}})$ a singleton projection of $\mathcal{I}$ on some element $d \in \Delta^{\mathcal{I}}$. Then $C^{\mathcal{J}} = C^{\mathcal{I}} \cap \Delta^{\mathcal{J}}$.*

Corollary 1. *Let $\alpha = C \sqsubseteq D$ be a $\mathcal{P}$-axiom, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation such that $\mathcal{I} \models \alpha$, and $\mathcal{J}$ a singleton projection of $\mathcal{I}$ on an element $d \in \Delta^{\mathcal{I}}$. Then $\mathcal{J} \models \alpha$.*

Corollary 2. *Let $\alpha = C \sqsubseteq D$ be a $\mathcal{P}$-axiom, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation such that $\mathcal{I} \not\models \alpha$. Then there exists $d \in \Delta^{\mathcal{I}}$ such that for the singleton projection $\mathcal{J}$ of $\mathcal{I}$ to d, $\mathcal{J} \not\models \alpha$.*

Given an ontology network $\mathcal{N}$, a *singleton model agreement* for $\mathcal{N}$ is a model agreement μ such that for every $\mathcal{O}$ in $\mathcal{N}$ every interpretation $\mathcal{I} \in \mu(\mathcal{O})$ is a singleton.

Lemma 19. *Let $\mathcal{N}$ be a $\mathcal{P}$-ontology network, $\mathcal{O}$ an ontology in $\mathcal{N}$, and α a $\mathcal{P}$-axiom such that $\mathcal{O} \not\models_{\mathcal{N}} \alpha$. Then there exists a singleton model agreement μ' for $\mathcal{N}$ and a model $\mathcal{I}' \in \mu'(\mathcal{O})$ such that $\mathcal{I}' \not\models \alpha$.*

Proof. Since $\mathcal{O} \not\models_{\mathcal{N}} \alpha$, there exists a model agreement μ over a domain Δ such that for some $\mathcal{I} \in \mu(\mathcal{O})$ we have $\mathcal{I} \not\models \alpha$. By Corollary 2, there exists an element $d \in \Delta$ such that for the singleton projection $\mathcal{I}'$ of $\mathcal{I}$ to d, we have $\mathcal{I}' \not\models \alpha$. Now define a mapping μ' by setting $\mu'(\mathcal{O})$, for every ontology $\mathcal{O}$ in $\mathcal{N}$, to consist of the singleton projections of the interpretations in $\mu(\mathcal{O})$ to d. By Corollary 1, we have $\mathcal{I}' \models \mathcal{O}$, for all $\mathcal{I}' \in \mu'(\mathcal{O})$. To prove that μ' is a model agreement it remains to show that if $\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle \in \mathcal{N}$ and $\mathcal{I}'_1 \in \mu'(\mathcal{O}_1)$ then there exists $\mathcal{I}'_2 \in \mu'(\mathcal{O}_2)$ such that $\mathcal{I}'_1 =_\Sigma \mathcal{I}'_2$. Indeed, since $\mathcal{I}'_1 \in \mu'(\mathcal{O}_1)$ then $\mathcal{I}'_1$ is a singleton projection of some $\mathcal{I}_1 \in \mu(\mathcal{O}_1)$. Since μ is a model agreement, there exists $\mathcal{I}_2 \in \mu(\mathcal{O}_2)$ such that $\mathcal{I}_1 =_\Sigma \mathcal{I}_2$. Let $\mathcal{I}'_2$ be the singleton projection of $\mathcal{I}_2$ to d. Then $\mathcal{I}'_2 \in \mu'(\mathcal{O}_2)$ by definition of μ'. Furthermore, since $\mathcal{I}_1 =_\Sigma \mathcal{I}_2$, it is easy to see by the Definition 3 that $\mathcal{I}'_1 =_\Sigma \mathcal{I}'_2$. $\square$

Lemma 19 implies, in particular, that to check the entailment in $\mathcal{P}$ networks, it is sufficient to restrict attention only to singleton model agreements. W.l.o.g., one can assume that these singleton model agreements have the same domain. Similarly, only interpretation of symbols that appear in $\mathcal{N}$ or in the checked axiom α counts. Since the number of interpretations of concept names over one element domain is at most exponential in the number of concept names, for checking entailment $\mathcal{O} \models_{\mathcal{N}} \alpha$ it is sufficient to restrict attention to only exponentially-many singleton model agreements in the size of $\mathcal{N}$ and α. This gives a simple $\mathsf{NExpTime}$ algorithm for checking whether $\mathcal{O} \not\models_{\mathcal{N}} \alpha$: guess a singleton model agreement μ (of an exponential size), and check whether $\mathcal{I} \not\models \alpha$ for some $\mathcal{I} \in \mu(\mathcal{O})$. It is, however, possible to find the required model agreement deterministically, thereby reducing the complexity to $\mathsf{ExpTime}$.

Theorem 12. *There is an $\mathsf{ExpTime}$ procedure that given a $\mathcal{P}$-ontology network $\mathcal{N}$, an ontology $\mathcal{O}$ in $\mathcal{N}$, and a $\mathcal{P}$-axiom α, checks whether $\mathcal{O} \models_{\mathcal{N}} \alpha$.*

Proof. Let Σ be the set of all signature symbols appearing in $\mathcal{N}$ and α. Since $\mathcal{N}$ and α are formulated in $\mathcal{P}$, Σ consists of only concept names. Let d be a fixed (domain) element. For every subset $s \subseteq \Sigma$, let $\mathcal{I}(s) = (\{d\}, \cdot^{\mathcal{I}(s)})$ be a singleton interpretation defined by $A^{\mathcal{I}(s)} = \{d\}$ if $A \in s$ and $A^{\mathcal{I}(s)} = \emptyset$ otherwise. Finally, let m be a mapping defined by $m(\mathcal{O}) = 2^{\Sigma}$ for all ontologies $\mathcal{O}$ in $\mathcal{N}$. Clearly, the mapping m can be constructed in exponential time in the size of $\mathcal{N}$.

The mapping m corresponds to the assignment μ of singleton interpretations to ontologies in $\mathcal{N}$ defined as $\mu(\mathcal{O}) = \{\mathcal{I}(s) \mid s \in m(\mathcal{O})\}$. This assignment, however, is not necessarily a model agreement for $\mathcal{N}$ according to Definition 2. First, not all interpretations $\mathcal{I}(s)$ for $s \in m(\mathcal{O})$ are models of $\mathcal{O}$. Second, it is not guaranteed that for every $\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle \in \mathcal{N}$ and every $s_1 \in m(\mathcal{O}_1)$ there exists $s_2 \in m(\mathcal{O}_2)$ such that $\mathcal{I}(s_1) =_\Sigma \mathcal{I}(s_2)$. To fix the defects of the first type, we remove from $m(\mathcal{O})$ all sets s such that $\mathcal{I}(s) \not\models \mathcal{O}$. (It is easy to check in polynomial time if a singleton interpretation is a model of an ontology). To fix the defects of the second type, we remove all $s_1 \in m(\mathcal{O}_1)$ for which there exists no $s_2 \in m(\mathcal{O}_2)$ such that $s_1 \cap \Sigma = s_2 \cap \Sigma$. We repeat performing this operation until no defects are left.

Clearly, both operations can be performed in exponential time in $\mathcal{N}$ since there are at most exponentially-many values s that can be removed. Finally, to decide whether $\mathcal{O} \models_{\mathcal{N}} \alpha$, we check whether $\mathcal{I}(s) \models \alpha$ for all $s \in m(\mathcal{O})$. If this property holds, we return $\mathcal{O} \models_{\mathcal{N}} \alpha$; otherwise, we return $\mathcal{O} \not\models_{\mathcal{N}} \alpha$.

We claim that the above algorithm is correct. Indeed, if $\mathcal{O} \not\models_{\mathcal{N}} \alpha$ is returned then there is a model agreement μ and an interpretation $\mathcal{I} \not\models \alpha$ such that $\mathcal{I} \in \mu(\mathcal{O})$.

Conversely, if $\mathcal{O} \not\models_{\mathcal{N}} \alpha$ then there exists a model agreement μ for $\mathcal{N}$ such that $\mathcal{I} \not\models \alpha$ for some $\mathcal{I} \in \mu(\mathcal{O})$. Then by Lemma 19, there exists a singleton model agreement μ' for $\mathcal{N}$ such that $\mathcal{I}' \not\models \alpha$ for some $\mathcal{I}' \in \mu'(\mathcal{O})$. For a singleton interpretation $\mathcal{I}$, let $s(\mathcal{I}) = \{A \mid A^{\mathcal{I}} \neq \emptyset\}$. W.l.o.g., $\mathcal{I}(s(\mathcal{I}')) = \mathcal{I}'$ for each $\mathcal{I}' \in \mu'(\mathcal{O})$, with $\mathcal{O}$ in $\mathcal{N}$. Let m' be a mapping defined by $m'(\mathcal{O}) = \{s(\mathcal{I}) \mid \mathcal{I} \in \mu'(\mathcal{O})\}$ for each $\mathcal{O}$ in $\mathcal{N}$. So $\mathcal{I}(s) \in \mu'(\mathcal{O})$ for every $s \in m'(\mathcal{O})$ and $\mathcal{O}$ in $\mathcal{N}$. By induction over the construction of m, it is easy to show that $m'(\mathcal{O}) \subseteq m(\mathcal{O})$ for every $\mathcal{O}$ in $\mathcal{N}$. Indeed, since $m(\mathcal{O})$ is initialized with all subsets of the signature Σ, $m'(\mathcal{O}) \subseteq m(\mathcal{O})$ holds in the beginning. Furthermore, for each $s \in m'(\mathcal{O})$, we have $\mathcal{I}(s) \in \mu'(\mathcal{O})$, and so, $\mathcal{I}(s) \models \mathcal{O}$. Thus, s cannot be removed from $m(\mathcal{O})$ as a defect of the first type. Similarly, for every $\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle \in \mathcal{N}$ and every $s_1 \in m'(\mathcal{O}_1)$, we have $\mathcal{I}_1 = \mathcal{I}(s_1) \in \mu'(\mathcal{O}_1)$.

Since μ' is a model agreement, there exists $\mathcal{I}_2 \in \mu'(\mathcal{O}_2)$ such that $\mathcal{I}_1 =_\Sigma \mathcal{I}_2$. Hence for $s_2 = s(\mathcal{I}_2) \in m'(\mathcal{O}_2)$, we have $s_1 \cap \Sigma = s_2 \cap \Sigma$. Therefore, s_1 cannot be removed from $m(\mathcal{O}_1)$ as a defect of the second type. Finally, since $\mathcal{I}' \not\models \alpha$ for some $\mathcal{I}' \in \mu'(\mathcal{O})$, we have $s' = s(\mathcal{I}') \in m'(\mathcal{O}) \subseteq m(\mathcal{O})$. Hence, our algorithm returns $\mathcal{O} \not\models_\mathcal{N} \alpha$. □

It is possible to improve the upper bound obtained in Theorem 12 for acyclic $\mathcal{P}$-ontology networks.

Theorem 13. *There is a* PSpace *procedure that given an acyclic $\mathcal{P}$-ontology network $\mathcal{N}$, an ontology $\mathcal{O}$ in $\mathcal{N}$, and a $\mathcal{P}$-axiom α, checks whether $\mathcal{O} \models_\mathcal{N} \alpha$.*

Proof. As in the proof of Theorem 12, let Σ be the set of all signature symbols appearing in $\mathcal{N}$ and α, and d a fixed (domain) element. For each $s \subseteq \Sigma$, let $\mathcal{I}(s) = (\{d\}, \cdot^{\mathcal{I}}(s))$ be a singleton interpretation defined by $A^{\mathcal{I}(s)} = \{d\}$ if $A \in s$ and $A^{\mathcal{I}(s)} = \emptyset$ otherwise.

We describe a recursive procedure $P(\mathcal{O}, s)$ that given an ontology $\mathcal{O}$ in $\mathcal{N}$ and $s \subseteq \Sigma$ returns *true* if there exists a model agreement μ for $\mathcal{N}$ such that $\mathcal{I}(s) \in \mu(\mathcal{O})$, and returns *false* otherwise. The procedure works as follows. If $\mathcal{I}(s) \not\models \mathcal{O}$, $P(\mathcal{O}, s)$ return *false*. Otherwise, we iterate over all $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$ and for every $s' \subseteq \Sigma$ such that $s \cap \Sigma = s' \cap \Sigma$ and run $P(\mathcal{O}', s')$ recursively. If for each $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$ some of the recursive call $P(\mathcal{O}', s')$ returned *true*, we return *true* for $P(\mathcal{O}, s)$. Otherwise, we return *false*.

Clearly, $P(\mathcal{O}, s)$ always terminates since $\mathcal{N}$ is acyclic. Furthermore, the procedure can be implemented in polynomial space in the size of $\mathcal{N}$ and α, since the recursion depth is bounded by the size of $\mathcal{N}$ and at every recursive call, only the input values $\mathcal{O}$ and s need to be saved (assuming the iterations over the import relations and subsets of Σ use some fixed order).

Next we show that our procedure is correct. Assume that $P(\mathcal{O}, s)$ returns *true* for some input $\mathcal{O}$ and $s \subseteq \Sigma$. For each ontology $\mathcal{O}$ in $\mathcal{N}$, let $m(\mathcal{O})$ be the set of all $s \subseteq \Sigma$ such that there was a (recursive) call $P(\mathcal{O}, s)$ with the output *true*. Let μ be a mapping defined by $\mu(\mathcal{O}) = \{\mathcal{I}(s) \mid s \in m(\mathcal{O})\}$. We claim that μ is a model agreement for $\mathcal{N}$.

Indeed, since $P(\mathcal{O}, s)$ returns *true* only if $\mathcal{I}(s) \models \mathcal{O}$, for every $\mathcal{O}$ in $\mathcal{N}$ and $\mathcal{I} \in \mu(\mathcal{O})$ we have $\mathcal{I} \models \mathcal{O}$. Furthermore, if $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$ and $\mathcal{I} \in \mu(\mathcal{O})$ then, by definition of μ, there exists $s \in m(\mathcal{O})$ such that $\mathcal{I} = \mathcal{I}(s)$ and $P(s, \mathcal{O})$ returned *true*. In particular, since $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$, there was a recursive call $P(s', \mathcal{O}')$ that returned *true* for some $s' \subseteq \Sigma$ such that $s \cap \Sigma = s' \cap \Sigma$. By the definition of m, this means that $s' \in m(\mathcal{O}')$. Therefore, $\mathcal{I} =_\Sigma \mathcal{I}(s') \in \mu(\mathcal{O}')$, by the definition of μ. Thus, μ is a model agreement for $\mathcal{N}$.

Conversely, assume that there exists a model agreement μ for $\mathcal{N}$ such that $\mathcal{I}(s) \in \mu(\mathcal{O})$. Since $\mathcal{N}$ is a $\mathcal{P}$-ontology network, w.l.o.g., for every $\mathcal{O}$ in $\mathcal{N}$ and every $\mathcal{I} \in \mu(\mathcal{O})$, there exists $s \subseteq \Sigma$ such that $\mathcal{I} = \mathcal{I}(s)$. We prove that $P(\mathcal{O}, s)$ returns *true* for every $\mathcal{O}$ and s such that $\mathcal{I}(s) \in \mu(\mathcal{O})$. Indeed, assume to the contrary that $P(\mathcal{O}, s)$ returns *false* for some s such that $\mathcal{I}(s) \in \mu(\mathcal{O})$ and when executing $P(\mathcal{O}, s)$, there was no other recursive call to $P(\mathcal{O}', s')$ that returned *false* for some s' such that $\mathcal{I}(s') \in \mu(\mathcal{O}')$. Since $\mathcal{I}(s) \in \mu(\mathcal{O})$, we have $\mathcal{I}(s) \models \mathcal{O}$, thus $P(\mathcal{O}, s)$ cannot return *false* due to $\mathcal{I}(s) \not\models \mathcal{O}$. Hence there exists $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$ such that for every $s' \subseteq \Sigma$ with $s \cap \Sigma = s' \cap \Sigma$, the recursive call of $P(\mathcal{O}', s')$ returned *false*. Then $\mathcal{I}(s') \notin \mu(\mathcal{O}')$ for all such s' by our assumption above. Since μ is a model agreement, $\langle \mathcal{O}, \Sigma, \mathcal{O}' \rangle \in \mathcal{N}$, and $\mathcal{I}(s) \in \mu(\mathcal{O})$, there exists $\mathcal{I}' \in \mu(\mathcal{O}')$ such that $\mathcal{I}(s) =_\Sigma \mathcal{I}'$. Then $\mathcal{I}' = \mathcal{I}(s')$ for some s' such that $s \cap \Sigma = s' \cap \Sigma$. This gives us a contradiction since $\mathcal{I}(s') \in \mu(\mathcal{O}')$ for no s', with $s \cap \Sigma = s' \cap \Sigma$.

Now, to check whether $\mathcal{O} \models_\mathcal{N} \alpha$ using the procedure P, we enumerate all $s \subseteq \Sigma$ and check whether $\mathcal{I}(s) \not\models \alpha$ and $P(\mathcal{O}, s)$ returns *true*. We return $\mathcal{O} \not\models_\mathcal{N} \alpha$ if such s exists, and $\mathcal{O} \models_\mathcal{N} \alpha$ otherwise. This algorithm is correct. Indeed, if such s exists, then there exists a model agreement μ for $\mathcal{N}$ such that $\mathcal{I}(s) \in \mu(\mathcal{O})$. Hence, $\mathcal{O} \not\models_\mathcal{N} \alpha$. Conversely, if $\mathcal{O} \not\models_\mathcal{N} \alpha$ then there exists a model agreement μ for $\mathcal{N}$ and some $\mathcal{I} \in \mu(\mathcal{O})$ such that $\mathcal{I} \models \mathcal{O}$ and $\mathcal{I} \not\models \alpha$. By Lemma 19, w.l.o.g. one can assume that μ is a singleton model agreement. Then there exists some $s \subseteq \Sigma$ such that $\mathcal{I}(s) \in \mu(\mathcal{O})$ and $\mathcal{I}(s) \not\models \alpha$. Since $\mathcal{I}(s) \in \mu(\mathcal{O})$, $P(\mathcal{O}, s)$ should return *true*. Hence, our algorithm returns $\mathcal{O} \not\models_\mathcal{N} \alpha$. □

9 Conclusions

We have introduced a new mechanism for ontology integration which is based on semantic import relations between ontologies and is a generalization of the standard OWL importing. In order to import an external ontology $\mathcal{O}$ into a local one, one has to specify an import relation, which defines a set of symbols, whose semantics should be borrowed from $\mathcal{O}$. The significant feature of the proposed mechanism, which comes natural in complex ontology integration scenarios, is that every ontology has its own view on ontologies it refines and the views on the same ontology are independent unless coordinated by import relations. We have shown that this feature can lead to an exponential increase of the time complexity of reasoning over ontologies combined with acyclic import relations. Intuitively, this is because one has to consider multiple views of the same ontology, each of which gives a different ontology. When cyclic importing is allowed, the complexity jumps to undecidability, even if every ontology in a combination is given in the DL $\mathcal{EL}$. Similarly, this is because one has to consider infinitely many views on the same ontology. These complexity results are shown for situations when the imported symbols include roles. It is natural to ask whether the complexity drops when the imported symbols are concept names. The second parameter which may influence the complexity of reasoning is the semantics which is 'imported'. In the proposed mechanism, importing the semantics of symbols is implemented via agreement of models of ontologies. One can consider refinements of this mechanism, e.g., by carefully selecting the classes of models of ontologies which must be agreed. The third way to decrease the complexity is to restrict the language in which ontologies are formulated. We conjecture that reasoning with cyclic imports is decidable for ontologies formulated in the family of DL-Lite.

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Appendix

Proofs for Section 6

Theorem 1. *Entailment in acyclic $\mathcal{EL}$-ontology networks is ExpTime-hard.*

Let $M = \langle Q, \mathcal{A}, \delta\rangle$ be a TM and $n = \mathtt{1exp}(m)$ an exponential, for $m \geqslant 0$. Consider an ontology $\mathcal{O}$ defined for M and n by axioms (29)-(31) below:

$$A \sqsubseteq \exists r^{n\cdot(2n+3)}.(\mathtt{q}_0 \sqcap \exists(r,\mathtt{b})^{2n+2}) \tag{29}$$

where $A \notin Q \cup \mathcal{A}$.

$$\exists r^{2n}(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W, \tag{30}$$

for all $X, Y, U, Z, W \in Q \cup \mathcal{A}$ such that $XYUZ \overset{\delta'}{\mapsto} W$.

$$\exists r.\mathtt{q_h} \sqsubseteq H,\ \ \exists r.H \sqsubseteq H \tag{31}$$

Lemma 20. *M accepts the empty word in n steps iff $\mathcal{O} \models A \sqsubseteq H$.*

Proof. For the purpose of this proof, we let *configuration* of M be a word of length $4n+3$ in the alphabet $Q \cup \mathcal{A}$. Then, given a configuration $\mathfrak{c}$, the notion of successor configuration is naturally induced by δ'. Let us call the word of the form

$$\mathfrak{c}_0 = \underbrace{\mathtt{b}\ldots\mathtt{b}}_{2n}\ \mathtt{q}_0\ \underbrace{\mathtt{b}\ldots\mathtt{b}}_{2n+2}$$

initial configuration of M.

Then M accepts the empty word in n steps iff there is a sequence $\mathfrak{c}_0, \ldots, \mathfrak{c}_n$ of configurations in the above sense such that for all $0 \leqslant i < n$, $\mathfrak{c}_{i+1}$ is a successor of $\mathfrak{c}_i$ and $\mathtt{q_h}$ is the state symbol in $\mathfrak{c}_n$.

Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ be a model of ontology $\mathcal{O}$ with a domain element $a \in A^{\mathcal{I}}$. Then by axiom (29), there is a r-chain outgoing from a which contains $n+1$ consequent segments $s_0, \ldots, s_n$ of length $2n+3$. We consider each segment as a linearly ordered set of elements from Δ and for $1 \leqslant j \leqslant 2n+3$, we denote by $s_i[j]$ the j-th element of s_i. Given a word w of length $2n+3$, we say that segment s_i *represents* w if $s_i[j] \in w[j]^{\mathcal{I}}$, for all $1 \leqslant j \leqslant 2n+3$. We assume the following enumeration of segments in the r-chain:

$$\underbrace{\ldots}_{s_n}\ \cdots\ \underbrace{\ldots}_{s_1}\underbrace{\mathtt{q}_0\mathtt{b}\ldots\mathtt{b}}_{s_0}$$

i.e. we let s_0 represent a fragment of the initial configuration.

We show that for all $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2n+3$, it holds $s_i[j] \in (\mathfrak{c}_i[2n+j-i])^{\mathcal{I}}$. Then clearly, for every $0 \leqslant i \leqslant n$, there exists j such that $s_i[j] \in \mathtt{q}^{\mathcal{I}}$, where $\mathtt{q}$ is the state symbol from $\mathfrak{c}_i$. In particular, $s_n[j] \in \mathtt{q_h}^{\mathcal{I}}$, for some j, and hence $a \in H^{\mathcal{I}}$, due to axiom (31). We use induction on

i. The case $i = 0$ is obvious, so let us assume that the claim holds for $0 \leqslant i < n$. Note that configuration $\mathfrak{c}_i$ has the form

$$\underbrace{\mathtt{b}\ldots\mathtt{b}}_{\geqslant 2n-i}\ldots\mathtt{q}\ldots\underbrace{\mathtt{b}\ldots\mathtt{b}}_{\geqslant 2n+2-i}$$

thus, by the induction hypothesis, s_i represents a fragment w_i of $\mathfrak{c}_i$ having the form

$$\ldots\mathtt{q}\ldots\underbrace{\mathtt{b}\ldots\mathtt{b}}_{\geqslant 2n+2-2i}$$

where $w_i[j] = \mathfrak{c}_i[2n+j-i]$, for all $1 \leqslant j \leqslant 2n+3$. Note that since $i < n$, we have $2n+2-2i \geqslant 4$.

Then by axiom (30), s_{i+1} must represent a word w_{i+1} of the form

$$\underbrace{\ldots}_{3}\ \underbrace{\ldots\ldots\ \underbrace{\mathtt{b}\ldots\ldots\ldots\mathtt{b}}_{\geqslant 2n+2-2(i+1)\geqslant 2}}_{2n}$$

where for $4 \leqslant j \leqslant 2n+3$ and $k = j-3$, it holds

$$w_i[k]\ w_i[k+1]\ w_i[k+2]\ w_i[k+3] \overset{\delta'}{\mapsto} w_{i+1}[j].$$

Since we have $w_i[j] = \mathfrak{c}_i[2n+j-i]$, for $1 \leqslant j \leqslant 2n+3$, it follows by definition of δ' that $w_{i+1}[j] = \mathfrak{c}_{i+1}[2n+j-(i+1)]$, for $4 \leqslant j \leqslant 2n+3$. It remains to show how the first three symbols in w_{i+1} are defined.

By axiom (30), it holds

$$w_{i+1}[2n+3]\ w_i[1]\ w_i[2]\ w_i[3] \overset{\delta'}{\mapsto} w_{i+1}[3]$$

$$w_{i+1}[2n+2]\ w_{i+1}[2n+3]\ w_i[1]\ w_i[2] \overset{\delta'}{\mapsto} w_{i+1}[2]$$

$$w_{i+1}[2n+1]\ w_{i+1}[2n+2]\ w_{i+1}[2n+3]\ w_i[1] \overset{\delta'}{\mapsto} w_{i+1}[1]$$

and we have $w_{i+1}[2n+2] = w_{i+1}[2n+3] = \mathtt{b}$. By the induction hypothesis, $w_i[k] = \mathfrak{c}_i[2n+k-i]$, for $k = 1, 2, 3$, hence, $\mathfrak{c}_i[2n-i] = \mathfrak{c}_i[2n-i-1] = \mathtt{b}$ and therefore, $w_{i+1}[j] = \mathfrak{c}_i[2n+j-(i+1)]$, for $j = 2, 3$. Note that at most one of $w_{i+1}[2n+1]$, $w_i[1]$ is a state symbol, because otherwise $n = 1$, which is not the case, since we have $n = \mathtt{1exp}(m)$ and $m \geqslant 1$. Hence, we conclude that $w_{i+1}[1] = \mathfrak{c}_i[2n+1-(i+1)]$.

For the 'if' direction, suppose M does not accept the empty word in n steps. Consider an interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ having domain $\Delta = \{x_1, \ldots x_k\}$, for $k = (n+1)\cdot(2n+3)$, such that:

- $r^{\mathcal{I}} = \{\langle x_i, x_{i+1}\rangle\}_{1 \leqslant i < k}$;
- $A^{\mathcal{I}} = \{x_1\}$ and $H^{\mathcal{I}} = \mathtt{q_h}^{\mathcal{I}} = \varnothing$;
- $\mathtt{q_0}^{\mathcal{I}} = \{x_p\}$ and $x_l \in \mathtt{b}^{\mathcal{I}}$, for $p = n\cdot(2n+3)+1$ and $p+1 \leqslant l \leqslant k$;
- for any $1 \leqslant i \leqslant n\cdot(2n+3)$ and $W \in Q \cup \mathcal{A}$, it holds $x_i \in W^{\mathcal{I}}$ iff there exist $V_0, \ldots, V_3 \in Q \cup \mathcal{A}$ such that $x_{i+2n+j} \in V_j$, for $0 \leqslant j \leqslant 3$, and $V_0\ V_1\ V_2\ V_3 \overset{\delta'}{\mapsto} W$.

By using arguments from the proof of the 'only if' direction, one can verify that $\mathcal{I}$ is well defined and $\mathcal{I}$ is a model of ontology $\mathcal{O}$ such that $\mathcal{I} \not\models A \sqsubseteq H$. $\square$

To complete the proof of the theorem let us show that ontology $\mathcal{O}$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. Note that $\mathcal{O}$ contains axioms (29), (30) with concepts of size exponential in m. Consider axiom (29) and a concept inclusion φ of the form

$$A \sqsubseteq \exists r^{n\cdot(2n+3)}.B$$

where B is a concept name. Observe that it is equivalent to

$$A \sqsubseteq \underbrace{\exists r^p.\exists r^p}_{2\text{ times}}.\underbrace{\exists r^n.\exists r^n.\exists r^n}_{3\text{ times}}.B$$

where $p = \mathtt{1exp}(2m)$. Consider axiom ψ of the form $A \sqsubseteq B$. By iteratively applying Lemma 13 we obtain that $\psi[B \mapsto \exists r^p.\exists r^p.B]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. By repeating this argument we obtain that the same holds for φ. Further, by Lemma 2 the axiom $\theta = \varphi[B \mapsto \mathtt{q_0} \sqcap B]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. Again, by iteratively applying Lemma 13 together with Lemma 2 we conclude that $\theta[B \mapsto \exists(r,\mathtt{b})^{2n+2}]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m and hence, so is axiom (29). The expressibility of axioms of the form (30) is shown identically. The remaining axioms of ontology $\mathcal{O}$ are $\mathcal{EL}$-axioms whose size does not depend on m. By applying Lemma 1 we obtain that there exists an acyclic $\mathcal{EL}$-ontology network $\mathcal{N}$ of size polynomial in m and an ontology $\mathcal{O}_{\mathcal{N}}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible and thus, it holds $\mathcal{O}_{\mathcal{N}} \models_{\mathcal{N}} A \sqsubseteq H$ iff M accepts the empty word in $\mathtt{1exp}(m)$ steps. Theorem 1 is proved.

Theorem 2. *Entailment in cyclic $\mathcal{EL}$-ontology networks is RE-hard.*

For a TM $M = \langle Q, \mathcal{A}, \delta\rangle$, we define an infinite ontology $\mathcal{O}$, which contains variants of axioms (29-30) from Theorem 1 and additional axioms for a correct implementation of transitions of M. Ontology $\mathcal{O}$ consists of the following families of axioms:

$$A \sqsubseteq \exists r^k.(\exists v^l.L \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists(r,\mathtt{b})^{2l+2})) \tag{32}$$

for $A, L, \varepsilon \notin Q \cup \mathcal{A}$ and all $k, l \geqslant 0$;

$$\exists r.\exists v^k.L \sqsubseteq \exists v^k.L, \quad k \geqslant 0 \tag{33}$$

$$\exists v^k.L \sqcap \exists r^{2k+4}.\varepsilon \sqsubseteq \varepsilon, \quad k \geqslant 0 \tag{34}$$

$$\exists v^k.L \sqcap \tag{35}$$
$$\sqcap \exists r^{2k+1}.(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W$$

for $k \geqslant 0$ and all $X, Y, U, Z, W \in Q \cup \mathcal{A}$ such that $XYUZ \overset{\delta'}{\mapsto} W$;

$$\exists v^k.L \sqcap \qquad (36)$$
$$\sqcap \exists r^{2k}.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W$$

for $k \geqslant 0$ and all $Y, U, Z, W \in Q \cup \mathcal{A}$ such that $\mathtt{b}YUZ \overset{\delta'}{\mapsto} W$;

$$\exists v^k.L \sqcap \qquad (37)$$
$$\sqcap \exists r^{2k}.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W$$

for $k \geqslant 0$ and all $U, Z, W \in Q \cup \mathcal{A}$ such that $\mathtt{bb}UZ \overset{\delta'}{\mapsto} W$;

$$\exists v^k.L \sqcap \qquad (38)$$
$$\sqcap \exists r^{2k}.(X \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.Z)))) \sqsubseteq W$$

for $k \geqslant 0$ and all $X, Z, W \in Q \cup \mathcal{A}$ such that $X\mathtt{bb}Z \overset{\delta'}{\mapsto} W$;

$$\mathtt{q_h} \sqsubseteq H, \ \ \exists r.H \sqsubseteq H \qquad (39)$$

Lemma 21. *It holds $\mathcal{O} \models A \sqsubseteq H$ iff M halts.*

Proof. Suppose M halts in n steps; w.l.o.g. we assume that $n > 1$. Let $\mathcal{I}$ be a model of $\mathcal{O}$ with a domain element $a \in A^{\mathcal{I}}$. Then by axioms (32)-(35), $\mathcal{I}$ is a model of the concept inclusions:

$$A \sqsubseteq \exists r^{n\cdot(2n+4)}.(\exists v^n.L \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists (r, \mathtt{b})^{2n+2}))$$

$$\exists r.\exists v^n.L \sqsubseteq \exists v^n.L \qquad (48)$$

$$\exists v^n.L \sqcap \exists r^{2n+4}.\varepsilon \sqsubseteq \varepsilon \qquad (49)$$

$$\exists v^n.L \sqcap \qquad (50)$$
$$\sqcap \exists r^{2n+1}.(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W$$

for all $X, Y, U, Z, W \in Q \cup \mathcal{A}$ such that $XYUZ \overset{\delta'}{\mapsto} W$;

$$\exists v^n.L \sqcap \qquad (51)$$
$$\sqcap \exists r^{2n}.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W$$

for all $Y, U, Z, W \in Q \cup \mathcal{A}$ such that $\mathtt{b}YUZ \overset{\delta'}{\mapsto} W$;

$$\exists v^n.L \sqcap \qquad (52)$$
$$\sqcap \exists r^{2n}.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W$$

for all $U, Z, W \in Q \cup \mathcal{A}$ such that $\mathtt{bb}UZ \overset{\delta'}{\mapsto} W$;

$$\exists v^n.L \sqcap \qquad (53)$$
$$\sqcap \exists r^{2n}.(X \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.Z)))) \sqsubseteq W$$

for all $X, Z, W \in Q \cup \mathcal{A}$ such that $X\mathtt{bb}Z \overset{\delta'}{\mapsto} W$.

Hence, $\mathcal{I}$ gives a r-chain outgoing from a, which contains $n+1$ consequent segments $s_0, \dots s_n$ of length $2n+3$ separated by elements from the interpretation of ε. We use conventions and notations from the 'only-if' part of the proof of Theorem 1 and assume the following enumeration of segments in the r-chain:

$$\varepsilon \underbrace{\dots}_{s_n} \varepsilon \dots \varepsilon \underbrace{\dots}_{s_1} \varepsilon \underbrace{\mathtt{q_0 b} \dots \mathtt{b}}_{s_0}$$

i.e. we let s_0 represent a fragment of the initial configuration $\mathfrak{c}_0$ of M.

If M halts in n steps then there is a sequence $\mathfrak{c}_0, \dots, \mathfrak{c}_n$ of configurations such that for all $0 \leqslant i < n$, $\mathfrak{c}_{i+1}$ is a successor of $\mathfrak{c}_i$ and $\mathtt{q_h}$ is the state symbol in $\mathfrak{c}_n$. We show that for all $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2n+3$, it holds $s_i[j] \in (\mathfrak{c}_i[2n+j-i])^{\mathcal{I}}$. Then clearly, for every $0 \leqslant i \leqslant n$, there exists j such that $s_i[j] \in \mathtt{q}^{\mathcal{I}}$, where $\mathtt{q}$ is the state symbol from $\mathfrak{c}_i$. In particular, $s_n[j] \in \mathtt{q_h}^{\mathcal{I}}$, for some j, and hence $a \in H^{\mathcal{I}}$, due to axiom (39). We use induction on i. The case $i = 0$ is obvious, so let us assume that the claim holds for $0 \leqslant i < n$. By repeating the arguments from the proof of Theorem 1 one can verify that due to axioms (48)-(53), s_{i+1} must represent a word w_{i+1} of the form

$$\underbrace{\dots}_{3} \cdots\cdots \underbrace{\mathtt{b} \dots\dots\dots \mathtt{b}}_{\geqslant 2n+2-2(i+1) \geqslant 2}$$

where $w_{i+1}[j] = \mathfrak{c}_{i+1}[2n+j-(i+1)]$, for all $4 \leqslant j \leqslant 2n+3$, and $w_{i+1}[2n+2] = w_{i+1}[2n+3] = \mathtt{b}$. Note that $\mathcal{I}$ is a model of axioms (36)-(38) for $y = n$ and thus it holds:

$$w_{i+1}[2n+3]\ w_i[1]\ w_i[2]\ w_i[3] \overset{\delta'}{\mapsto} w_{i+1}[3]$$
$$w_{i+1}[2n+2]\ w_{i+1}[2n+3]\ w_i[1]\ w_i[2] \overset{\delta'}{\mapsto} w_{i+1}[2]$$
$$w_{i+1}[2n+1]\ w_{i+1}[2n+2]\ w_{i+1}[2n+3]\ w_i[1] \overset{\delta'}{\mapsto} w_{i+1}[1]$$

By the induction hypothesis, we have $w_i[k] = \mathfrak{c}_i[2n+k-i]$, for $k = 1, 2, 3$, so $\mathfrak{c}_i[2n-i] = \mathfrak{c}_i[2n-i-1] = \mathtt{b}$ and therefore, $w_{i+1}[j] = \mathfrak{c}_i[2n+j-(i+1)]$, for $j = 2, 3$. Note that at most one of $w_{i+1}[2n+1]$, $w_i[1]$ is a state symbol, because otherwise $n = 1$, which is not the case, since we have assumed $n > 1$. Hence, we conclude that $w_{i+1}[1] = \mathfrak{c}_i[2n+1-(i+1)]$.

For the 'if' direction, suppose that M does not halt. Consider interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ having an infinite domain Δ, which is a union of sets $R_{m,n} = \{x_0^{m,n}, \dots, x_{m+2n+3}^{m,n}\}$ and $V_t = \{y_1^t, \dots, y_t^t\}$, for all $m, n \geqslant 0$, $t \geqslant 1$. A set $R_{m,n}$ will be used to define a r-chain with a prefix of length $m+1$ representing fragments of consequent configurations of M and a postfix of length $2n+3$ representing a fragment of the initial configuration. The elements of V_t will be used to define a v-chain of length t indicating the length of a configuration fragment. More precisely, we define $\mathcal{I}$ as an interpretation satisfying the following properties:

- there is an element $a \in \Delta$ such that $\{a\} = A^{\mathcal{I}}$ and $a = x_0^{m,n}$, for all $m, n \geqslant 0$;
- the sets $R_{m,n} \setminus \{a\}$ and V_t, for $m, n \geqslant 0$, $t \geqslant 1$, are pairwise disjoint;
- $r^{\mathcal{I}} = \bigcup\{\langle x_i^{m,n}, x_{i+1}^{m,n}\rangle \mid 0 \leqslant i < m+2n+3,\ m,n \geqslant 0\}$;
- $v^{\mathcal{I}} = \bigcup\{\langle y_i^t, y_{i+1}^t\rangle \mid 1 \leqslant i < t,\ t \geqslant 1\} \cup \{\langle x_i^{m,n}, y_1^n\rangle \mid 0 \leqslant i \leqslant m,\ m \geqslant 0, n \geqslant 1\}$;
- $L^{\mathcal{I}} = \{a\} \cup \{x_i^{m,0}\}_{0 \leqslant i \leqslant m} \cup \{y_t^t\}_{t \geqslant 1}$.

Then one can readily verify that $\mathcal{I}$ is a model of axioms (33). Now let us define interpretation of ε and the alphabet symbols from $Q \cup \mathcal{A}$ as follows. Let ε, $\mathtt{q}_0$, and $\mathtt{b}$ be interpreted in $\mathcal{I}$ as:

- $\varepsilon^{\mathcal{I}} = \{x_m^{m,n} \mid m,n \geqslant 0\} \cup \{x_i^{m,n} \mid i = m - k(2n+4),\ m,n \geqslant 0,\ k \geqslant 1\}$;
- $\mathtt{q}_0 = \{x_i^{m,n} \mid i = m+1,\ m,n \geqslant 0\}$;
- $x_i^{m,n} \in \mathtt{b}^{\mathcal{I}}$, for $m+2 \leqslant i \leqslant m+2n+3$ and $m,n \geqslant 0$.

Then clearly, $\mathcal{I}$ is a model of axioms (32) and (34).

Now, for $0 \leqslant i \leqslant m$, $m,n \geqslant 0$, and $W \in Q \cup \mathcal{A}$, set $x_i^{m,n} \in W^{\mathcal{I}}$ iff there exist $W_0, \ldots, W_4 \in Q \cup \mathcal{A} \cup \{\varepsilon\}$ such that $x_{i+2n+j}^{m,n} \in W_j$, for $0 \leqslant j \leqslant 4$, and either of the following holds:

- $W_1\ W_2\ W_3\ W_4 \overset{\delta'}{\mapsto} W$;
- $W_0 = \mathtt{b}$, $W_1 = \varepsilon$, and $W_0\ W_2\ W_3\ W_4 \overset{\delta'}{\mapsto} W$;
- $W_0 = W_1 = \mathtt{b}$, $W_2 = \varepsilon$, and $W_0\ W_1\ W_3\ W_4 \overset{\delta'}{\mapsto} W$;
- $W_1 = W_2 = \mathtt{b}$, $W_3 = \varepsilon$, and $W_0\ W_1\ W_2\ W_4 \overset{\delta'}{\mapsto} W$.

It is not hard to verify that $\mathcal{I}$ defined in this way is a model of axioms (35)-(38).

Finally, let $H^{\mathcal{I}} = \mathtt{q_h}^{\mathcal{I}} = \varnothing$. Then by using arguments from the proof of the 'only if' direction, one can show that $\mathcal{I}$ is well defined and hence, $\mathcal{I}$ is a model of ontology $\mathcal{O}$ such that $\mathcal{I} \not\models A \sqsubseteq H$. ☐

To complete the proof of Theorem 2 we now show that ontology $\mathcal{O}$ is expressible by a cyclic $\mathcal{EL}$-ontology network. Let us demonstrate that so is the family of axioms (32). Let $\varphi = A \sqsubseteq B$ be a concept inclusion and $B, B1, B_2$ concept names. By Lemma 14, ontology $\mathcal{O}_1 = \{\varphi[B \mapsto \exists r^k.B] \mid k \geqslant 0\}$ is expressible by a cyclic $\mathcal{EL}$-ontology network. Then by Lemma 2, ontology $\mathcal{O}_2 = \mathcal{O}_1[B \mapsto B_1 \sqcap \varepsilon \sqcap \exists r.(\mathtt{q}_0 \sqcap B_2)]$ is expressible by a cyclic $\mathcal{EL}$-ontology network. By applying Lemma 14 again, we conclude that so is ontology $\mathcal{O}_3 = \bigcup_{l \geqslant 0} \mathcal{O}_2[B_1 \mapsto \exists v^l.B_1,\ B_2 \mapsto \exists (r, \mathtt{b})^{2l}.B_2]$, i.e., the ontology given by axioms

$$A \sqsubseteq \exists r^k.(\exists v^l.B_1 \sqcap \varepsilon \sqcap \exists r.(\mathtt{q}_0 \sqcap \exists (r,\mathtt{b})^{2l}.B_2))$$

for $k, l \geqslant 0$. Further, by Lemma 2, we obtain that $\mathcal{O}_2[B_1 \mapsto L,\ B_2 \mapsto \exists (r,\mathtt{b})^2]$ is expressible by a cyclic $\mathcal{EL}$-ontology network and hence, so is the family of axioms (32). A similar argument shows the expressibility of ontologies given by axioms (33)-(38). The remaining subset of axioms (39) of $\mathcal{O}$ is finite. By Lemma 1, there exists a cyclic $\mathcal{EL}$-ontology network $\mathcal{N}$ and an ontology $\mathcal{O}_{\mathcal{N}}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible and thus, it holds $\mathcal{O}_{\mathcal{N}} \models_{\mathcal{N}} A \sqsubseteq H$ iff M halts. Theorem 2 is proved.

Theorem 3. *Entailment in $\mathcal{ALC}$-ontology networks is 2ExpTime-hard.*

We prove the theorem by showing a reduction from the word problem for alternating Turing machines working with words of length $\mathtt{1exp}(n)$. For $n \geqslant 0$, let $M = \langle Q, \mathcal{A}, \delta_1, \delta_2\rangle$ be such ATM and let $\mathcal{O}$ be an ontology consisting of the following axioms, which implement a computation of M.

The first two axioms are used to initialize a r-chain (with the end marker E) used for 'storing' configurations of the ATM:

$$Z \sqsubseteq \exists (r, C)^{\mathtt{1exp}(n)}.\exists r.E \tag{54}$$

$$C \sqsubseteq \neg E \tag{55}$$

The next two axioms define a r-chain 'storing' the initial configuration of the form $\mathtt{b}\mathtt{q}_0\mathtt{b}\ldots\mathtt{b}$:

$$A \sqsubseteq Z \sqcap \forall r.\mathtt{b} \sqcap \forall r.\forall r.\mathtt{q}_0 \sqcap \forall r.\forall r.\forall r.B \tag{56}$$

$$B \sqsubseteq \mathtt{b} \sqcap \forall r.(E \sqcup B) \tag{57}$$

Axioms (58)-(59) initialize markers $S_\exists, S_\forall$ for configuration types and propagate them to the end of a r-chain:

$$q_\exists \sqsubseteq S_\exists,\ \ q_\forall \sqsubseteq S_\forall \tag{58}$$

for all $q_\exists \in Q_\exists$ and $q_\forall \in Q_\forall$;

$$\neg E \sqcap S_\exists \sqsubseteq \forall r.S_\exists,\ \ \neg E \sqcap S_\forall \sqsubseteq \forall r.S_\forall \tag{59}$$

Axioms (60)-(62) initialize labels C_1, C_2 to distinguish between successor configurations and enforce that every r-chain, which represents a $\exists$-configuration ($\forall$-configuration, respectively), has a subsequent r-chain (two subsequent r-chains, respectively) representing successor configuration(s):

$$E \sqcap S_\exists \sqsubseteq \exists r.(Z \sqcap C_1) \sqcup \exists r.(Z \sqcap C_2) \tag{60}$$

$$E \sqcap S_\forall \sqsubseteq \exists r.(Z \sqcap C_1) \sqcap \exists r.(Z \sqcap C_2) \tag{61}$$

$$C_\alpha \sqsubseteq \forall r.(E \sqcup C_\alpha),\quad \alpha = 1,2 \tag{62}$$

Axioms (63)-(66) initialize markers S_{XYUV} ($X, Y, U, V \in Q \cup \mathcal{A}$), which encode 4-tuples of symbols from configurations, while respecting the end points of r-chains:

$$X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.V)) \sqsubseteq \forall r.\forall r.S_{XYUV} \tag{63}$$

$$Z \sqcap \exists r.(U \sqcap \exists r.V) \sqsubseteq \forall r.S_{\mathtt{bb}UV} \tag{64}$$

$$Z \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.V)) \sqsubseteq \forall r.\forall r.S_{\mathtt{b}YUV} \tag{65}$$

$$X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.E)) \sqsubseteq \forall r.\forall r.S_{XYU\mathtt{b}} \tag{66}$$

for all $X, Y, U, V \in Q \cup \mathcal{A}$.

Finally, for $f(n) = \mathtt{1exp}(n) + 2$ and $\alpha = 1, 2$, axioms (67) implement transitions of M by initializing label concepts on the corresponding successor r-chains, and axiom (68) forbids the rejecting state:

$$S_{XYUV} \sqsubseteq \forall r^{f(n)}.(\neg C_\alpha \sqcup W) \tag{67}$$

for all $X, Y, U, V, W \in Q \cup \mathcal{A}$ such that $XYUV \overset{\delta_\alpha}{\mapsto} W$;

$$q_r \sqsubseteq \bot \tag{68}$$

Lemma 22. *It holds $\mathcal{O} \not\models A \sqsubseteq \bot$ iff M accepts the empty word.*

Proof. We assume that every configuration of M is a word of length $\mathtt{1exp}(n)$ in the alphabet $Q \cup \mathcal{A}$.

$(\Leftarrow)$: Let $\mathcal{C}$ be the set of configurations in an accepting run tree of M, with the root being the initial configuration $\mathfrak{c}_0 = \mathtt{b}\mathtt{q}_0\mathtt{b}\ldots\mathtt{b}$. We define a model of ontology $\mathcal{O}$, in which the interpretation of concept A is not empty. Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ be an interpretation, with $\Delta = \{x_{\mathfrak{c},i} \mid \mathfrak{c} \in \mathcal{C},\ 0 \leqslant i \leqslant \mathtt{1exp}(n) + 1\}$. Let us define $r^{\mathcal{I}} = \{\langle x_{\mathfrak{c},i}, x_{\mathfrak{c},i+1}\rangle \mid 0 \leqslant i \leqslant \mathtt{1exp}(n)\} \cup \{\langle x_{\mathfrak{c},\mathtt{1exp}(n)+1}, x_{\mathfrak{c}',0}\rangle \mid \mathfrak{c}'$ is a successor of $\mathfrak{c}\}$. Further, set $A^{\mathcal{I}} = \{x_{\mathfrak{c}_0,0}\}$, $Z^{\mathcal{I}} = \{x_{\mathfrak{c},0} \mid \mathfrak{c} \in \mathcal{C}\}$, $E^{\mathcal{I}} = \{x_{\mathfrak{c},\mathtt{1exp}(n)+1} \mid \mathfrak{c} \in \mathcal{C}\}$, $C_k^{\mathcal{I}} = \{x_{\mathfrak{c},i} \mid 0 \leqslant i \leqslant \mathtt{1exp}(n),\ \exists \mathfrak{c}' \in \mathcal{C}$ s.t. $\mathfrak{c}$ is a δ_α-successor of $\mathfrak{c}'\}$, for $\alpha = 1, 2$, and for all $X \in Q \cup \mathcal{A}$, set $X^{\mathcal{I}} = \{x_{\mathfrak{c},i} \mid \mathfrak{c}[i] = X\}$. For all $X, Y, U, V \in Q \cup \mathcal{A}$, concepts S_{XYUV} are interpreted in a clear way. Finally, let $C^{\mathcal{I}} = \{x_{\mathfrak{c},i} \mid 1 \leqslant i \leqslant \mathtt{1exp}(n)\}$, $B^{\mathcal{I}} = \{x_{\mathfrak{c}_0,i} \mid i \in \{1\} \cup \{3, \ldots, \mathtt{1exp}(n)\}\}$, $S_\exists^{\mathcal{I}} = \{x_{\mathfrak{c},i} \mid k \leqslant i \leqslant \mathtt{1exp}(n) + 1,\ \mathfrak{c}[k] \in Q_\exists\}$, $S_\forall^{\mathcal{I}} = \{x_{\mathfrak{c},i} \mid k \leqslant i \leqslant \mathtt{1exp}(n) + 1,\ \mathfrak{c}[k] \in Q_\forall\}$, and $q_r^{\mathcal{I}} = \varnothing$. Since every configuration from $\mathcal{C}$ is accepting, q_r appears in no $\mathfrak{c} \in \mathcal{C}$ and it is straightforward to verify that $\mathcal{I}$ is a model of $\mathcal{O}$.

$(\Rightarrow)$: Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ be a model of $\mathcal{O}$ and $x \in \Delta$ an element such that $x \in A^{\mathcal{I}}$. We say that a segment $x_1, \ldots x_{\mathtt{1exp}(n)+1}$ of a r-chain in $\mathcal{I}$ represents a configuration $\mathfrak{c} = uqw$ if $x_i \in \mathfrak{c}[i]$, for all $1 \leqslant i \leqslant \mathtt{1exp}(n)$, $x_{\mathtt{1exp}(n)+1} \in E^{\mathcal{I}}$, and $x_{\mathtt{1exp}(n)+1} \in S_\exists^{\mathcal{I}}$ if $q \in Q_\exists$ and $x_{\mathtt{1exp}(n)+1} \in S_\forall^{\mathcal{I}}$ if $q \in Q_\forall$. We show how to use $\mathcal{I}$ to define an accepting run tree of M in which for every configuration $\mathfrak{c}$ there is a segment in $\mathcal{I}$, which represents $\mathfrak{c}$. We use induction on the height of the accepting run tree. For the induction base, observe that by axioms (54)-(59), since $x \in A^{\mathcal{I}}$, there is a r-chain outgoing from x, which contains a segment representing the initial configuration $\mathfrak{c}_0$. We set $\mathfrak{c}_0$ to be the root of the tree. In the induction step, consider an arbitrary configuration $\mathfrak{c} = uqw$ being a leaf in the tree constructed so far. By the induction assumption, there is a segment $x_1, \ldots x_{\mathtt{1exp}(n)+1}$ in $\mathcal{I}$ which represents $\mathfrak{c}$. If $q \in Q_\exists$ then $x_{\mathtt{1exp}(n)+1} \in S_\exists^{\mathcal{I}}$ and by axioms (60), (54) there exist $y_0, \ldots y_{\mathtt{1exp}(n)+1} \in \Delta$ such that $\langle x_{\mathtt{1exp}(n)+1}, y_0\rangle \in r^{\mathcal{I}}$ and $\langle y_i, y_{i+1}\rangle \in r^{\mathcal{I}}$, for $0 \leqslant i \leqslant \mathtt{1exp}(n)$. Then by axioms (62)-(67), there is a successor configuration $\mathfrak{c}'$ of $\mathfrak{c}$ such that $y_i \in \mathfrak{c}'[i]$, for $1 \leqslant i \leqslant \mathtt{1exp}(n)$. Due to axiom (54) we have $y_{\mathtt{1exp}(n)+1} \in E^{\mathcal{I}}$ and by axioms (58)-(59) it holds $y_{\mathtt{1exp}(n)+1} \in S_\exists^{\mathcal{I}}$ if $q \in Q_\exists$ and $y_{\mathtt{1exp}(n)+1} \in S_\forall^{\mathcal{I}}$ if $q \in Q_\forall$, i.e., the segment $y_1, \ldots y_{\mathtt{1exp}(n)+1}$ represents $\mathfrak{c}'$. Since $\mathcal{I}$ is a model of axiom (68), $\mathfrak{c}'$ is an accepting configuration. We extend the tree by adding a child node $\mathfrak{c}'$ for the node $\mathfrak{c}$. Similarly, in case $q \in Q_\forall$ we have $x_{\mathtt{1exp}(n)+1} \in S_\forall^{\mathcal{I}}$ and hence by axioms (61)-(67), there exist two segments in $\mathcal{I}$, each representing an accepting successor configuration $\mathfrak{c}_i'$ of $\mathfrak{c}$, for $i = 1, 2$. Then we extend the tree by adding child nodes $\mathfrak{c}_i'$, $i = 1, 2$, for the node $\mathfrak{c}$. □

To conclude the proof of Theorem 3 let us show that ontology $\mathcal{O}$ is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n. Note that $\mathcal{O}$ contains axioms (54), (67) with concepts of size exponential in n. Note that by Lemma 4, axiom (54) is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in n. Now consider an axiom φ of the form (67)

$$S_{XYUV} \sqsubseteq \forall r^{\mathtt{1exp}(n)}.\forall r.\forall r.(\neg C_\alpha \sqcup W)$$

where $\alpha \in \{1, 2\}$, and a concept inclusion ψ defined as

$$S_{XYUV} \sqsubseteq D$$

where D is a concept name. By Lemma 12, $\psi[D \mapsto \forall r^{\mathtt{1exp}(n)}.\forall r.\forall r.(\neg C_\alpha \sqcup W)]$ is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n and hence, so is φ.

We conclude that each of axioms (54), (67) is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n. The remaining axioms of $\mathcal{O}$ are $\mathcal{ALC}$ axioms, whose size does not depend on n. Then by applying Lemma 1 we obtain that there exists an acyclic $\mathcal{ALC}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_{\mathcal{N}}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible and thus, it holds $\mathcal{O}_{\mathcal{N}} \models_{\mathcal{N}} A \sqsubseteq \bot$ iff M does not accept the empty word. Theorem 3 is proved.

Theorem 5. *Entailment in $\mathcal{ALCHOIF}$-ontology networks is coN2ExpTime-hard.*

The theorem is proved by a reduction from the complement of the (bounded) domino tiling problem. A *domino system* is a triple $\mathcal{D} = (T, V, H)$, where $T = \{1, \ldots, p\}$ is a finite set of tiles and $H, V \subseteq T \times T$ are horizontal and vertical matching relations. A *tiling* of size $m \times m$ for a domino system $\mathcal{D}$ with initial condition $c^0 = \langle t_1^0, \ldots, t_k^0\rangle$, where $t_i^0 \in T$, for $1 \leqslant i \leqslant k$, is a mapping $t : \{1, \ldots, m\} \times \{1, \ldots, m\} \to T$ such that $\langle t(i-1, j),\ t(i, j)\rangle \in V$, for $1 < i \leqslant m$, $1 \leqslant j \leqslant m$, $\langle t(i, j-1),\ t(i, j)\rangle \in H$, for $1 \leqslant i \leqslant m$, $1 < j \leqslant m$, and $t(1, j) = t_j^0$, for $1 \leqslant j \leqslant k$. It is well known that it is N2ExpTime-complete to decide whether a domino system admits a tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$, $n \geqslant 0$, with an initial condition c^0.

Let $\mathcal{D}$ be a domino system and $c^0 = \langle t_1^0, \ldots, t_n^0\rangle$ an initial condition. We define an ontology $\mathcal{O}$ consisting of the axioms, which encode the tiling problem for $\mathcal{D}$ with c^0 using a grid of dimension $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$, $n \geqslant 0$, to be 'tiled'. The first axiom of $\mathcal{O}$ defines the initial point of the grid, while the second one initializes a r-chain (having the end marker E and containing $\mathtt{1exp}(n) + 1$ many points) used to represent a sequence of $\mathtt{1exp}(n)$ many bits for a binary counter:

$$A \sqsubseteq Z \sqcap \forall r.(Z_v \sqcap Z_h) \tag{69}$$

$$Z \sqsubseteq \exists r^{\mathtt{1exp}(n)}.E \tag{70}$$

The next axioms set the bits given by X, Y on the r-chain outgoing from A to 'zero':

$$Z_v \sqsubseteq \neg X \sqcap \forall r.Z_v,\ Z_h \sqsubseteq \neg Y \sqcap \forall r.Z_h \tag{71}$$

The following three axioms define markers E_v and E_h, which 'hold' at the end of a r-chain, iff all the bits (except possibly the last bit) given by X, respectively by Y, are 1:

$$Z \sqsubseteq \forall r.(E_v \sqcap E_h) \tag{72}$$
$$E_v \sqcap X \sqsubseteq \forall r.E_v,\quad \neg(E_v \sqcap X) \sqsubseteq \forall r.\neg E_v \tag{73}$$
$$E_h \sqcap Y \sqsubseteq \forall r.E_h,\quad \neg(E_h \sqcap Y) \sqsubseteq \forall r.\neg E_h \tag{74}$$

The next axioms say that every r-chain 'containing' at least one zero bit has a 'vertical' and 'horizontal' successor r-chains:

$$E \sqcap \neg(E_v \sqcap X) \sqsubseteq \exists v.Z \tag{75}$$

$$E \sqcap \neg(E_h \sqcap Y) \sqsubseteq \exists h.Z \tag{76}$$

Axioms (77)-(78) initialize markers X^f, Y^f corresponding to the flipping conditions of a binary counter:

$$E \sqcup \exists r.(X \sqcap X^f) \sqsubseteq X^f,\ \exists r.\neg(X \sqcap X^f) \sqsubseteq \neg X^f \tag{77}$$
$$E \sqcup \exists r.(Y \sqcap Y^f) \sqsubseteq Y^f,\ \exists r.\neg(Y \sqcap Y^f) \sqsubseteq \neg Y^f \tag{78}$$

Axioms (79) define the role hierarchy needed for a correct incrementation of binary counters across role chains:

$$r \sqsubseteq v,\ \ r \sqsubseteq h \tag{79}$$

For $f(n) = \mathtt{1exp}(n) + 1$, axioms (80)-(83) define the values of bits in the vertical/horizontal successor r-chains given the values of the flipping markers:

$$X^f \sqsubseteq (X \sqcap \forall v^{f(n)}.\neg X) \sqcup (\neg X \sqcap \forall v^{f(n)}.X) \tag{80}$$

$$\neg X^f \sqsubseteq (X \sqcap \forall v^{f(n)}.X) \sqcup (\neg X \sqcap \forall v^{f(n)}.\neg X) \tag{81}$$

$$Y^f \sqsubseteq (Y \sqcap \forall h^{f(n)}.\neg Y) \sqcup (\neg Y \sqcap \forall h^{f(n)}.Y) \tag{82}$$

$$\neg Y^f \sqsubseteq (Y \sqcap \forall h^{f(n)}.Y) \sqcup (\neg Y \sqcap \forall h^{f(n)}.\neg Y) \tag{83}$$

The next four axioms enforce the grid structure composed by vertical and horizontal r-chains: for $f(n) = \mathtt{1exp}(n) + 1$, axioms (84)-(85) propagate the bit values given by X horizontally and those given by Y vertically; axiom (86) states that there is a unique common final element of the vertical and horizontal r-chains, in which all bits are 1; axiom (87) states that the roles v, h are inverse functional:

$$\top \sqsubseteq (X \sqcap \forall h^{f(n)}.X) \sqcup (\neg X \sqcap \forall h^{f(n)}.\neg X) \tag{84}$$
$$\top \sqsubseteq (Y \sqcap \forall v^{f(n)}.Y) \sqcup (\neg Y \sqcap \forall v^{f(n)}.\neg Y) \tag{85}$$

$$E \sqcap E_v \sqcap X \sqcap E_h \sqcap Y \sqsubseteq \{a\} \tag{86}$$

$$\mathtt{Fun}(v^-),\ \ \mathtt{Fun}(h^-) \tag{87}$$

Finally, axioms (88)-(95), where $f(n) = \mathtt{1exp}(n) + 1$, declare tile types $D_1, \ldots, D_p$ stating that there is a unique tile for every element labelled by E, and define tiling conditions as well as the initial tiling given by tile types $D_{t^0_k}$, $1 \leqslant k < m$:

$$E \sqsubseteq D_1 \sqcup \ldots \sqcup D_p \tag{88}$$
$$D_i \sqcap D_j \sqsubseteq \bot \qquad 1 \leqslant i < j \leqslant p \tag{89}$$
$$D_i \sqsubseteq \forall v^{f(n)}.D^v_i \sqcap \forall h^{f(n)}.D^h_i \qquad 1 \leqslant i \leqslant p \tag{90}$$
$$D_i \sqcap D^v_j \sqsubseteq \bot \qquad \langle i, j\rangle \notin V \tag{91}$$
$$D_i \sqcap D^h_j \sqsubseteq \bot \qquad \langle i, j\rangle \notin H \tag{92}$$
$$E \sqcap Z_v \sqcap Z_h \sqsubseteq I_1 \tag{93}$$
$$I_j \sqsubseteq \forall h^{f(n)}.I_{j+1} \qquad 1 \leqslant j < k \tag{94}$$
$$I_j \sqsubseteq D_{t^0_j} \qquad 1 \leqslant j \leqslant k \tag{95}$$

Let us prove two auxiliary lemmas. The first one shows that the axioms of $\mathcal{O}$ enforce that every model $\mathcal{I} \models O$, in which $A^{\mathcal{I}} \neq \emptyset$, contains a structure 'implementing' a grid of size $\mathtt{2exp}(n)\times\mathtt{2exp}(n)$. The second lemma demonstrates the reduction of the tiling problem to entailment from $\mathcal{O}$. Finally we show that $\mathcal{O}$ is expressible by an acyclic $\mathcal{ALCHOIF}$-ontology network of size polynomial in n, which proves the Theorem.

First, we introduce some auxiliary notations. For a natural number $m \geqslant 0$, we denote by $i[m]_2$ the value of the i-th bit in the binary representation of m. Let $(\Delta, \cdot^{\mathcal{I}})$ be an interpretation, r a role, and X, Y concept names. For $x, y \in \Delta$ and $k \geqslant 1$, the notation $x[r]^k y$ means that there is a sequence of elements $x_1, \ldots x_{k+1} \in \Delta$ such that $x_1 = y$, $x_{k+1} = x$, and $\langle x_{l+1}, x_l\rangle \in r^{\mathcal{I}}$, for $1 \leqslant l \leqslant k$. We say that an element $y \in \Delta$ *represents* a tuple $\langle i, j\rangle$, where $1 \leqslant i, j \leqslant m$, for $m = \mathtt{2exp}(n)$ and $n \geqslant 0$, if there is $x \in \Delta$ such that $x[r]^k y$, for $k = \mathtt{1exp}(n)$, i.e., there is a sequence of elements $x_1, \ldots x_{k+1} \in \Delta$ such that $x_1 = y$, $x_{k+1} = x$ and $\langle x_{l+1}, x_l\rangle \in r^{\mathcal{I}}$, for $1 \leqslant l \leqslant k$, and it holds:

$x_l \in X^{\mathcal{I}}$ iff $l[i]_2 = 1$ and $x_l \in Y^{\mathcal{I}}$ iff $l[j]_2 = 1$.

A subset $X \subseteq \Delta$ represents a tuple $\langle i, j\rangle$ if so does every element $x \in X$.

Lemma 23. *For any model $(\Delta, \cdot^{\mathcal{I}})$ of ontology $\mathcal{O}$ and any $x \in A^{\mathcal{I}}$ there exist elements $x_{i,j} \in \Delta$, $1 \leqslant i, j \leqslant \mathtt{2exp}(n)$ such that*

- $x_{i,j} \in E^{\mathcal{I}}$;
- $x_{1,1} \in (Z_v \sqcap Z_h)^{\mathcal{I}}$;

and for any $y \in \Delta$, it holds:

- $x[r]^{\mathtt{1exp}(n)} y$ *iff* $y = x_{1,1}$;
- $x_{i,j}[h]^{\mathtt{1exp}(n)+1} y$ *iff* $y = x_{i,j+1}$;
- $x_{i,j}[v]^{\mathtt{1exp}(n)+1} y$ *iff* $y = x_{i+1,j}$.

Proof. We use induction on $1 \leqslant i, j \leqslant \mathtt{2exp}(n)$ and define non-empty sets $X_{i,j}$ satisfying the following properties:

(a) $X_{1,1} = \{y \in \Delta \mid x[r]^{\texttt{1exp}(n)}y \text{ and } y \in (E \sqcap Z_v \sqcap Z_h)^{\mathcal{I}}\}$;

(b) every $X_{i,j}$ represents $\langle i, j\rangle$;

(c) $\forall x_{i,j} \in X_{i,j}$ it holds $x_{i,j} \in E^{\mathcal{I}}$ and $x_{i,j} \in (\neg(E_v \sqcap X))^{\mathcal{I}}$ iff $i < \texttt{2exp}(n)$ and $x_{i,j} \in (\neg(E_h \sqcap Y))^{\mathcal{I}}$ iff $j < \texttt{2exp}(n)$;

(d) $\forall x_{i-1,j} \in X_{i-1,j}\ \exists x_{i,j} \in X_{i,j}$ and $\forall x_{i,j} \in X_{i,j}\ \exists x_{i-1,j} \in X_{i-1,j}$ such that $x_{i-1,j}[v]^{\texttt{1exp}(n)}x_{i,j}$, when $i \geqslant 2$;

(e) $\forall x_{i,j-1} \in X_{i,j-1}\ \exists x_{i,j} \in X_{i,j}$ and $\forall x_{i,j} \in X_{i,j}\ \exists x_{i,j-1} \in X_{i,j-1}$ such that $x_{i,j-1}[h]^{\texttt{1exp}(n)}x_{i,j}$, when $j \geqslant 2$.

After that we show that the axioms of $\mathcal{O}$ enforce that every $X_{i,j}$ is a singleton, which proves the lemma. Initially we let every $X_{i,j}$ be equal to the empty set.

In the induction base, for $i = j = 1$, note that since $x \in A^{\mathcal{I}}$, by axioms (69)-(71), there is a sequence of elements $x_1, \ldots x_{k+1}$, with $k = \texttt{1exp}(n)$ and $x_1 = x$, such that $\langle x_m, x_{m+1}\rangle \in r^{\mathcal{I}}$, for $1 \leqslant m \leqslant k$, and $x_m \notin X^{\mathcal{I}} \cup Y^{\mathcal{I}}$, for $2 \leqslant m \leqslant k{+}1$, and $x_{k+1} \in (E \sqcap Z_v \sqcap Z_h)^{\mathcal{I}}$. We put every element x_{k+1} of such sequence into the set $X_{1,1}$. By definition, $X_{1,1}$ satisfies conditions (a),(b) and it follows from axioms (72)-(76) that $X_{1,1}$ satisfies condition (c) as well. Conditions (d),(e) are trivially satisfied, since $i = j = 1$.

In the induction step, for $3 \leqslant i + j \leqslant 2 \cdot \texttt{2exp}(n)$, we assume that the statement is proved for all sets $X_{i',j'}$, with $i' + j' < i + j$.

If $i \geqslant 2$, consider the set $X_{i-1,j}$. Since $i - 1 < \texttt{2exp}(n)$, by the induction assumption, for every $x \in X_{i-1,j}$, we have $x \in [E \sqcap \neg(E_v \sqcap X)]^{\mathcal{I}}$. Hence by axioms (75), (70), for every $x \in X_{i-1,j}$ there exists a sequence of elements $x_1, \ldots x_{k+1}$, with $k = \texttt{1exp}(n)$, such that $\langle x, x_1\rangle \in v^{\mathcal{I}}$, $\langle x_m, x_{m+1}\rangle \in r^{\mathcal{I}}$, for $1 \leqslant m \leqslant k$, and $x_{k+1} \in E^{\mathcal{I}}$. We put every element x_{k+1} into the set $X_{i,j}$. By axiom (79) and the definition of $X_{i,j}$, condition (d) holds for $X_{i,j}$. By the induction assumption, $X_{i-1,j}$ represents $\langle i-1, j\rangle$ and hence, by axioms (77), (79), (85), $X_{i,j}$ represents $\langle i, j\rangle$. i.e. $X_{i,j}$ satisfies condition (b). Moreover, it is easy to see that by axioms (72), (73), for every $x \in X_{i,j}$, it holds $x \in (\neg(E_v \sqcap X))^{\mathcal{I}}$ iff $i < \texttt{2exp}(n)$ and $x \in (\neg(E_h \sqcap Y))^{\mathcal{I}}$ iff $j < \texttt{2exp}(n)$, i.e., $X_{i,j}$ also satisfies (c).

Similarly, if $j \geqslant 2$, we consider the set $X_{i,j-1}$. Since $j - 1 < \texttt{2exp}(n)$, by the induction assumption, for every $x \in X_{i,j-1}$, it holds $x \in [E \sqcap \neg(E_h \sqcap X)]^{\mathcal{I}}$. Hence, by axioms (76), (70), for every $x \in X_{i,j-1}$ there exists a sequence of elements $x_1, \ldots x_{k+1}$, with $k = \texttt{1exp}(n)$, such that $\langle x, x_1\rangle \in h^{\mathcal{I}}$, $\langle x_m, x_{m+1}\rangle \in r^{\mathcal{I}}$, for $1 \leqslant m \leqslant k$, and $x_{k+1} \in E^{\mathcal{I}}$. We put every element x_{k+1} into the set $X_{i,j}$. By axiom (79) and the definition of $X_{i,j}$, condition (e) holds for $X_{i,j}$. By the induction assumption, $X_{i,j-1}$ represents $\langle i, j-1\rangle$ and hence, by axioms (78), (79), (84), $X_{i,j}$ represents $\langle i, j\rangle$, i.e. $X_{i,j}$ satisfies condition (b). Moreover, it is easy to see that by axioms (72), (74), for every $x \in X_{i,j}$, it holds $x \in (\neg(E_v \sqcap X))^{\mathcal{I}}$ iff $i < \texttt{2exp}(n)$ and $x \in (\neg(E_h \sqcap Y))^{\mathcal{I}}$ iff $j < \texttt{2exp}(n)$, i.e., $X_{i,j}$ also satisfies (c).

Now let us show by induction on $1 \leqslant i, j \leqslant \texttt{2exp}(n)$ that every set $X_{i,j}$ is a singleton. In the induction base for $i = j = \texttt{2exp}(n)$, by condition (c), for every $x \in X_{i,j}$ we have $x \in (E \sqcap E_v \sqcap X \sqcap E_h \sqcap Y)^{\mathcal{I}}$. Hence, it follows from axiom (86) that the set $X_{i,j}$ is a singleton. In the induction step we assume that the statement is shown for all $X_{i',j'}$, with $i' + j' > i + j$. If $i < \texttt{2exp}(n)$ then consider the set $X_{i+1,j}$. Since, $X_{i+1,j}$ satisfies property (d), it follows from the inverse functionality of v that $X_{i,j}$ is a singleton. If $j < \texttt{2exp}(n)$ then we consider the set $X_{i,j+1}$. Since, $X_{i,j+1}$ satisfies property (e), it follows from the inverse functionality of h that $X_{i,j}$ is a singleton. □

Lemma 24. *It holds $\mathcal{O} \not\models A \sqsubseteq \bot$ iff the domino system $\mathcal{D}$ admits a tiling of size $\texttt{2exp}(n) \times \texttt{2exp}(n)$.*

Proof. ($\Rightarrow$): Let $(\Delta, \cdot^{\mathcal{I}})$ be a model of ontology $\mathcal{O}$ and x an element such that $x \in A^{\mathcal{I}}$. Then there exist elements $x_{i,j} \in \Delta$, for $1 \leqslant i, j \leqslant \texttt{2exp}(n)$, having the properties as in Lemma 23. By axioms (88)-(89), for every $x_{i,j}$ there is a unique D_k, $1 \leqslant k \leqslant p$, such that $x_{i,j} \in D_k^{\mathcal{I}}$. By axioms (93)-(95) and the property of $x_{1,1}$ from Lemma 23, we have $x_{1,j} \in D_{t_j^0}$, for $1 \leqslant j \leqslant k$. Finally, axioms (90)-(92) enforce that for any elements $x_{i,j}$, $x_{i',j'}$, with $1 \leqslant i, j, i', j' \leqslant \texttt{2exp}(n)$, any D_k, D_l such that $1 \leqslant k, l \leqslant p$, $x_{i,j} \in D_k$, and $x_{i',j'} \in D_l$, we have $i' = i + 1$ iff $\langle k, l\rangle \in V$ and $j' = j + 1$ iff $\langle k, l\rangle \in H$. Therefore, we conclude that the domino system $\mathcal{D}$ admits a tiling.

($\Leftarrow$): For $m = \texttt{2exp}(n)$, let $t : \{1, \ldots, m\} \times \{1, \ldots, m\} \rightarrow T$ be a tiling for $\mathcal{D}$. We define a model $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ of $\mathcal{O}$, in which the interpretation of A is non-empty. Let $\Delta = \{x_{k,i,j} \mid 1 \leqslant k \leqslant \texttt{1exp}(n) + 1,\ 1 \leqslant i, j \leqslant \texttt{2exp}(n)\}$ and define interpretation of roles r, v, h as follows:

- $r^{\mathcal{I}} = \{\langle x_{k-1,i,j}, x_{k,i,j}\rangle \mid 2 \leqslant k\}$;
- $v^{\mathcal{I}} = r^{\mathcal{I}} \cup \{\langle x_{k,i-1,j}, x_{1,i,j}\rangle \mid k = \texttt{1exp}(n){+}1,\ 2 \leqslant i\}$;
- $h^{\mathcal{I}} = r^{\mathcal{I}} \cup \{\langle x_{k,i,j-1}, x_{1,i,j}\rangle \mid k = \texttt{1exp}(n) + 1,\ 2 \leqslant j\}$.

Further, we set $A^{\mathcal{I}} = \{x_{1,1,1}\}$, $Z^{\mathcal{I}} = \{x_{1,i,j} \mid 1 \leqslant i, j\}$, $E^{\mathcal{I}} = \{x_{k,i,j} \mid k = \texttt{1exp}(n){+}1\}$, $Z_v^{\mathcal{I}} = Z_h^{\mathcal{I}} = \{x_{k,1,1} \mid 2 \leqslant k \leqslant \texttt{1exp}(n) + 1\}$, $X^{\mathcal{I}} = \{x_{\texttt{1exp}(n)+2-k,i,j} \mid k[i]_2 = 1\}$, $Y^{\mathcal{I}} = \{x_{\texttt{1exp}(n)+2-k,i,j} \mid k[j]_2 = 1\}$, and $a^{\mathcal{I}} = x_{k,i,j}$, for $k = \texttt{1exp}(n) + 1$ and $i = j = \texttt{2exp}(n)$. Finally, we define $D_l^{\mathcal{I}} = \{x_{k,i,j} \mid k = \texttt{1exp}(n){+}1,\ t(i,j) = l\}$, for $1 \leqslant l \leqslant p$, and $I_j^{\mathcal{I}} = \{x_{i,1,j} \mid i = \texttt{1exp}(n) + 1\}$, for $1 \leqslant j \leqslant k$. Other concepts X^f, Y^f, E_v, E_h, and D_i^v, D_i^h, for $1 \leqslant i \leqslant p$, are interpreted in a clear way. It is straightforward to verify that $\mathcal{I}$ is a model of every axiom of $\mathcal{O}$. □

To complete the proof of Theorem 5 let us show that ontology $\mathcal{O}$ is expressible by an acyclic $\mathcal{ALCHOIF}$-ontology network of size polynomial in n. Note that $\mathcal{O}$ contains axioms (70), (80)-(85), (90) with concepts of size exponential in n. By Lemma 4, axiom (70) is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in n. Now consider an axiom φ of the form (80):

$$X^f \sqsubseteq (X \sqcap \forall v^{\texttt{1exp}(n)}.\forall v.\neg X) \sqcup (\neg X \sqcap \forall v^{\texttt{1exp}(n)}.\forall v.X)$$

Let ψ be concept inclusion of the form

$$X^f \sqsubseteq (X \sqcap \bar{B}) \sqcup (\neg X \sqcap B)$$

where $\bar{B}, B$ are concept names. By Lemma 12, $\psi[\bar{B} \mapsto \forall v^{\mathtt{1exp}(n)}.\forall v.\neg X,\ B \mapsto \forall v^{\mathtt{1exp}(n)}.\forall v.X]$ is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n and hence, so is φ. The expressibility of axioms of the form (82)-(85), and (90) is proved analogously.

We conclude that each of the axioms (70), (80)-(85), (90)-(92) is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n. The remaining axioms of $\mathcal{O}$ are $\mathcal{ALCHOIF}$ axioms, whose size does not depend on n. Then by applying Lemma 1 we obtain that there exists an acyclic $\mathcal{ALCHOIF}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff the domino system $\mathcal{D}$ does not admit a tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$. Theorem 5 is proved.

Theorem 7. *Entailment in $\mathcal{H}$-Networks is* ExpTime*-hard.*

Proof. We show that the word problem for ATMs working with words of a polynomial length n reduces to entailment in cyclic $\mathcal{H}$-ontology networks. Then, since APSpace = ExpTime, the claim follows.

Let $M = \langle Q, \mathcal{A}, \delta_1, \delta_2 \rangle$ be a ATM. We call the word of the form $\mathtt{b}\mathtt{q}_0\mathtt{b}\ldots\mathtt{b}$ *initial configuration* $\mathfrak{c}_{\mathtt{init}}$ of M.

Consider signature σ consisting of concept names B_{ai}, for $a \in Q \cup \mathcal{A}$ and $1 \leqslant i \leqslant n$ (with the informal meaning that the i-th symbol in a configuration of M is a). Let σ^1, and σ^2 be 'copies' of signature σ consisting of the above mentioned concept names with the superscripts ^{1} and ^{2}, respectively.

For $\alpha = 1, 2$, let $\mathcal{O}^\alpha$ be an ontology consisting of the following axioms:

$$B^\alpha_{Xi-2} \sqcap B^\alpha_{Yi-1} \sqcap B^\alpha_{Ui} \sqcap B^\alpha_{Vi+1} \ \sqsubseteq\ B_{Wi} \tag{40}$$

for $3 \leqslant i \leqslant n-1$ and all $X, Y, U, V, W \in Q \cup \mathcal{A}$ such that $XYUV \overset{\delta_\alpha}{\mapsto} W$;

$$B^\alpha_{U1} \sqcap B^\alpha_{V2} \ \sqsubseteq\ B_{W1} \tag{41}$$

for all $U, V, W \in Q \cup \mathcal{A}$ such that $\mathtt{bb}UV \overset{\delta_\alpha}{\mapsto} W$;

$$B^\alpha_{Y1} \sqcap B^\alpha_{U2} \sqcap B^\alpha_{V3} \ \sqsubseteq\ B_{W2} \tag{42}$$

for all $Y, U, V, W \in Q \cup \mathcal{A}$ such that $\mathtt{b}YUV \overset{\delta_\alpha}{\mapsto} W$;

$$B^\alpha_{Xn-2} \sqcap B^\alpha_{Yn-1} \sqcap B^\alpha_{Un} \ \sqsubseteq\ B_{Wn} \tag{43}$$

for all $X, Y, U, W \in Q \cup \mathcal{A}$ such that $XYU\mathtt{b} \overset{\delta_\alpha}{\mapsto} W$;

$$B_{\mathtt{q_{rej}}i} \sqsubseteq \bar{H}, \quad \bar{H} \ \sqsubseteq\ \bar{H}^\alpha \tag{44}$$

for $1 \leqslant i \leqslant n$.

Let $\mathcal{O}$ be an ontology consisting of the following axioms:

$$\bar{H}^1 \sqcap B_{q_\forall i} \ \sqsubseteq\ \bar{H},\ \ \bar{H}^2 \sqcap B_{q_\forall i} \sqsubseteq \bar{H} \qquad \bar{H}^1 \sqcap \bar{H}^2 \sqcap B_{q_\exists i} \ \sqsubseteq\ \bar{H} \tag{45}$$

for $1 \leqslant i \leqslant m$, $\mathtt{q}_\exists \in Q_\exists$, and $\mathtt{q}_\forall \in Q_\forall$;

$$A \sqsubseteq \textstyle\bigsqcap_{1 \leqslant i \leqslant n+2} B_{\mathtt{b}i} \sqcap B_{q_0 n+3} \sqcap \bigsqcap_{n+4 \leqslant i \leqslant m} B_{\mathtt{b}i} \tag{46}$$

$$B_{ai} \sqsubseteq B^\alpha_{ai} \tag{47}$$

for $\alpha = 1, 2$, $1 \leqslant i \leqslant m$, and all $a \in Q \cup \mathcal{A}$.

Consider ontology network $\mathcal{N}$ consisting of the import relations $\langle \mathcal{O}, \Sigma^\alpha, \mathcal{O}^\alpha \rangle$ and $\langle \mathcal{O}^\alpha, \Sigma, \mathcal{O} \rangle$, where $\Sigma^\alpha = \{\bar{H}^\alpha\} \cup \sigma^\alpha$, $\Sigma = \{\bar{H}\} \cup \sigma$, and $\alpha = 1, 2$.

We claim that M does not accept the empty word iff $\mathcal{O} \models_\mathcal{N} A \sqsubseteq \bar{H}$.

($\Rightarrow$): Let $\mathfrak{c}$ be a configuration of M, $\mathcal{I}$ an interpretation, and x a domain element. We say that $\mathcal{I}$ *x-represents* $\mathfrak{c}$, if x belongs to the interpretation of the concept $\bigsqcap_{1 \leqslant i \leqslant m} B_{\mathfrak{c}[i]i}$. We show by induction that for any k-rejecting configuration $\mathfrak{c}$ and any model $\mathcal{I} \models_\mathcal{N} \mathcal{O}$, if $\mathcal{I}$ x-represents $\mathfrak{c}$, then $x \in \bar{H}^\mathcal{I}$. Then it follows that $\mathcal{O} \models_\mathcal{N} A \sqsubseteq \bar{H}$, whenever M does not accept the empty word, because every model $\mathcal{I} \models_\mathcal{N} \mathcal{O}$ x-represents $\mathfrak{c}_{\mathtt{init}}$, for $x \in A^\mathcal{I}$, due to axiom (46).

Let q be the state symbol in $\mathfrak{c}$. In the induction base $k = 0$, we have $q = \mathtt{q_{rej}}$ and hence, by the first axiom in (44), it holds $x \in \bar{H}^\mathcal{I}$. In the induction step $k \geqslant 1$, by the definition of network $\mathcal{N}$, for $\alpha = 1, 2$, there exists a model $\mathcal{J}_\alpha \models_\mathcal{N} \mathcal{O}^\alpha$, which agrees with $\mathcal{I}$ on Σ^α. Then by axioms (47), (40)-(43), for $\alpha = 1, 2$, there is a model $\mathcal{I}_\alpha \models_\mathcal{N} \mathcal{O}$, which agrees with $\mathcal{J}_\alpha$ on Σ and x-represents a successor configuration $\mathfrak{c}_\alpha$ of $\mathfrak{c}$ wrt δ_α. If $q \in Q_\forall$ then, since $\mathfrak{c}$ is k-rejecting, at least one of $\mathfrak{c}_1$, $\mathfrak{c}_2$ is $(k-1)$–rejecting and thus, by the induction assumption, we must have $x \in \bar{H}^{\mathcal{I}_\alpha}$, for some $\alpha = 1, 2$. Then by the second axiom in (44), it holds $x \in (\bar{H}^\alpha)^{\mathcal{I}_\alpha}$ and hence, $x \in (\bar{H}^\alpha)^{\mathcal{J}_\alpha}$, for some $\alpha = 1, 2$. Then $x \in \bar{H}^\mathcal{I}$, by the first two axioms in (45). The case $q \in Q_\exists$ is considered similarly.

($\Leftarrow$): Assume M accepts the empty word. Let $\mathfrak{c}$ be a configuration of M and let $\mathcal{I}$ be a singleton interpretation with a domain element x. We say that $\mathcal{I}$ *represents* configuration $\mathfrak{c}$ if $\mathcal{I}$ x-represents $\mathfrak{c}$. We show that there exists a singleton model agreement μ for $\mathcal{N}$ and a model $\mathcal{I} \in \mu(\mathcal{O})$ such that $\mathcal{I} \not\models A \sqsubseteq H$. By induction on $k \geqslant 0$ we define families of singleton interpretations $\{\mathcal{F}^k\}_{k \geqslant 0}$ and $\{\mathcal{F}^{\alpha,k}\}_{k \geqslant 0}$, for $\alpha = 1, 2$, having the following properties:

(a) for all $\mathcal{I} \in \mathcal{F}^k$ and $\mathcal{J}_\alpha \in \mathcal{F}^{\alpha,k}$, $k \geqslant 0$ and $\alpha = 1, 2$, it holds $\mathcal{I} \models \mathcal{O}$ and $\mathcal{J}_\alpha \models \mathcal{O}^\alpha$;

(b) for any $\mathcal{I} \in \mathcal{F}^k$ and $k \geqslant 0$, there exists $\mathcal{J}_\alpha \in \mathcal{F}^{\alpha,k}$, where $\alpha = 1, 2$, such that $\mathcal{I} =_{\Sigma^\alpha} \mathcal{J}_\alpha$;

(c) for any $\mathcal{J}_\alpha \in \mathcal{F}^{\alpha,k}$, $k \geqslant 0$, and $\alpha = 1, 2$, there exists $\mathcal{I}_\alpha \models \mathcal{F}^{k+1}$ such that $\mathcal{J}_\alpha =_\Sigma \mathcal{I}_\alpha$;

(d) any $\mathcal{I} \in \mathcal{F}^k$, $k \geqslant 0$, represents a configuration $\mathfrak{c}$ of M and it holds $(\bar{H}^\alpha)^\mathcal{I} \neq \varnothing$ iff either the state symbol in $\mathfrak{c}$ is $\mathtt{q_{rej}}$ or $\mathfrak{c}$ has a rejecting successor configuration wrt δ_α;

In the induction base, let $\mathcal{F}^0$ consist of a singleton interpretation $\mathcal{I}$, which represents $\mathfrak{c}_{\mathtt{init}}$ and has the properties:

(i) $A^{\mathcal{I}} \neq \varnothing$, $\bar{H}^{\mathcal{I}} = \varnothing$;

(ii) for all $a \in Q \cup \mathcal{A}$, $1 \leqslant i \leqslant n$, and $\alpha = 1, 2$, the interpretations of (B_{ai}) and B_{ai}^{α} coincide;

(iii) for all $\alpha = 1, 2$, the interpretation of $\bar{H}^{\alpha}$ is not empty iff $\mathfrak{c}_{\mathtt{init}}$ has a rejecting successor configuration wrt δ_{α}.

Clearly, $\mathcal{I}$ is a model of axioms (46),(47) and since $\mathfrak{c}_{\mathtt{init}}$ is an accepting configuration, $\mathcal{I}$ is a model of axioms (45), which means that $\mathcal{I} \models \mathcal{O}$.

In the induction step for $k \geqslant 1$, take an arbitrary model $\mathcal{I} \in \mathcal{F}^{k-1}$. Let $\mathcal{I}$ represent a configuration $\mathfrak{c}$ and let $\mathfrak{c}_{\alpha}$ be a successor configuration of $\mathfrak{c}$ wrt δ_{α}, for $\alpha = 1, 2$ (in the case, when $\mathfrak{c}$ does not have a successor configuration wrt δ_{α}, we set $\mathfrak{c}_{\alpha} = \mathfrak{c}$, for $\alpha = 1, 2$).

For $\alpha = 1, 2$, let $\mathcal{J}_{\alpha}$ be an interpretation which agrees with $\mathcal{I}$ on Σ^{α} and has the following properties:

- $\mathcal{J}_{\alpha}$ represents $\mathfrak{c}_{\alpha}$;
- $\bar{H}^{\mathcal{J}_{\alpha}} = (\bar{H}^{\alpha})^{\mathcal{I}}$.

One can readily verify that $\mathcal{J}_{\alpha}$ is a model of axioms (40)-(43). By the induction assumption, $\mathcal{I}$ has property (d), hence, $\mathcal{J}_{\alpha}$ is a model of axioms (44) and therefore, $\mathcal{J}_{\alpha} \models \mathcal{O}^{\alpha}$. For $\alpha = 1, 2$, let $\mathcal{I}_{\alpha}$ be an interpretation, which agrees with $\mathcal{J}_{\alpha}$ on Σ and has the following properties:

- $A^{\mathcal{I}_{\alpha}} = \varnothing$;
- $\mathcal{I}_{\alpha}$ represents $\mathfrak{c}_{\alpha}$;
- $\mathcal{I}_{\alpha}$ satisfies (ii);
- $(\bar{H}^{\alpha})^{\mathcal{I}_{\alpha}} \neq \varnothing$ iff either the state symbol in $\mathfrak{c}$ is $\mathtt{q_{rej}}$ or $\mathfrak{c}_{\alpha}$ has a rejecting successor configuration wrt δ_{α}.

Clearly, $\mathcal{I}_{\alpha}$, $\alpha = 1, 2$, is a model of axioms (46)-(47). Since $\mathcal{J}_{\alpha} =_{\Sigma} \mathcal{I}_{\alpha}$, we have $\bar{H}^{\mathcal{I}_{\alpha}} = \bar{H}^{\mathcal{J}_{\alpha}}$. Then by the definition of interpretation $\bar{H}^{\mathcal{J}_{\alpha}}$ it follows that $\mathcal{I}_{\alpha}$ is a model of axioms (45) and thus, $\mathcal{I}_{\alpha} \models \mathcal{O}$.

For all $\alpha = 1, 2$ and $k \geqslant 0$, let $\mathcal{F}^{\alpha,k}$ be the family of interpretations $\mathcal{J}_{\alpha}$ defined for a model $\mathcal{I} \in \mathcal{F}^{k}$, as described above. Similarly, let $\mathcal{F}^{k+1}$ be the family of interpretations $\mathcal{I}_{\alpha}$ defined for a model $\mathcal{J}_{\alpha} \in \mathcal{F}^{\alpha,k}$, for some $\alpha = 1, 2$, as above. By definition, the families of interpretations $\{\mathcal{F}^{k}\}_{k \geqslant 0}$ and $\{\mathcal{F}^{\alpha,k}\}_{k \geqslant 0}$, $\alpha = 1, 2$, have properties (a)-(c). Then a mapping μ defined as $\mu(\mathcal{O}) = \bigcup_{k \geqslant 0} \mathcal{F}^{k}$, $\mu(\mathcal{O}^{\alpha}) = \bigcup_{k \geqslant 0} \mathcal{F}^{\alpha,k}$, for $\alpha = 1, 2$, is a model agreement for $\mathcal{N}$ and there is a model $\mathcal{I} \in \mathcal{F}^{0} \subseteq \mu(\mathcal{O})$ such that $\mathcal{I} \not\models A \sqsubseteq \bar{H}$, which means that $\mathcal{O} \not\models_{\mathcal{N}} A \sqsubseteq \bar{H}$. $\square$