Decomposing Description Logic Ontologies

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Abstract

Recent years have seen the advent of large and complex ontologies, most notably in the medical domain. As a consequence, structuring mechanisms for ontologies are nowadays viewed as an indispensible tool. A basic such mechanism is the automatic decomposition of the vocabulary of an ontology into independent parts. In this paper, we study decompositions that are syntax independent in the sense that the resulting partitioning depends only on the meaning of the vocabulary items, but not on the concrete syntactic form of the axioms in the ontology.

We present the first systematic investigation of decompositions of this type in the context of ontologies. Specifically, we focus on ontologies formulated in description logics and provide a variety of results that range from theorems stating the existence of unique finest decompositions to complexity results and algorithms computing decompositions. We also investigate the relationship between the existence of unique finite decompositions and a variant of the Craig interpolation property called parallel interpolation.

Introduction

The purpose of an ontology in knowledge representation is to fix the vocabulary of an application domain and to formally describe the meaning of this vocabulary using a logic-based language. This simple idea has proved to be rather successful, and consequently a considerable number of ontologies have been developed for various application domains. In broad domains such as medicine, ontologies used in practice can be extremely large; as an example, take the medical ontology SNOMED CT that covers almost half a million vocabulary items. Unsurprisingly, the design and maintenance of logical theories of this size poses serious challenges and it has long been a major goal of the KR community to provide support in the form of automated reasoning techniques.

Basic reasoning support for ontology design and maintenance aims to make explicit the structure of an ontology, for example by using classification (computing the subconcept/superconcept hierarchy). This is fundamental for an ontology designer who can easily lose track of the overall structure of an ontology—especially when it is constructed by multiple designers working in parallel as in the case of SNOMED CT. Making explicit the structure is also essential when an existing ontology has to be re-engineered due to changes in the modeled application domain or to customize it for a novel application—especially when the ontology was designed by somebody else.

In this paper, we consider a way of analyzing the structure of an ontology that aims at making explicit the *dependencies* among vocabulary items in the ontology. Our approach is based on *signature decompositions*, a partition of the signature of an ontology (i.e., of the symbols used to describe vocabulary items) into parts that are independent regarding their meaning. Similar kinds of structural analysis of an ontology have been advocated, e.g. in (d'Aquin et al. 2009). However, all existing approaches are *syntax-dependent* in the sense that two semantically equivalent, but syntactically different ontologies may yield different decompositions. Thus, the quality of the computed signature decomposition depends on the quality of the representation of the analyzed ontology (when the goal of the analysis may actually be to improve the quality of a poorly organized ontology).

Our aim is to establish the theoretical foundations for a purely semantic approach to signature decompositions that is not syntax-dependent in the above sense. Formally, the basic notion studied in this paper is the following: a partition $\Sigma_1, \dots, \Sigma_n$ of the signature of an ontology $\mathcal{T}$ formulated in an ontology language $\mathcal{L}$ is a *signature decomposition* of $\mathcal{T}$ in $\mathcal{L}$ if there are ontologies $\mathcal{T}_1, \dots, \mathcal{T}_n$ formulated in $\mathcal{L}$ such that (i) each $\mathcal{T}_i$ uses only symbols from Σ_i and (ii) the union $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$ is logically equivalent to $\mathcal{T}$. This notion has first been proposed by Parikh (1999) and Kourousias and Makinson (2007) in the context of propositional logic and belief revision. We emphasize that the ontologies $\mathcal{T}_1, \dots, \mathcal{T}_n$ used in the definition of signature decompositions need *not* be subsets of the original ontology $\mathcal{T}$. Moreover, as we are interested in decompositions of signatures, we only demand the existence of these ontologies, but do not insist they are explicitly computed. There is a close relationship between signature decompositions and approaches to modularization of ontologies that aim at a partition of the *axioms* (rather than signature) of an ontology into independent parts (Cuenca Grau et al. 2006; Amir and McIlraith 2005; Stuckenschmidt, Parent, and Spaccapietra 2009). Again, however, all existing approaches are syntax-dependent and aim at partitioning the existing axiomatization.

In many cases, the initial version of signature decompositions defined above can be expected to be too coarse to be informative. To see this, consider a description logic (DL) ontology $\mathcal{T}$ that consists of the axioms $\alpha = (\mathsf{Car} \sqsubseteq \exists \mathsf{has\_part}.\mathsf{Tire})$ and $\beta = (\mathsf{Ship} \sqsubseteq \exists \mathsf{has\_part}.\mathsf{Deck})$. It is not difficult to show that, due to the use of the role has_part, the only decomposition of $\mathcal{T}$ consists of only one set that contains the whole signature. From an ontology design perspective, though, the ontology $\mathcal{T}$ contains cars and ships as two separate subject areas that should not be 'merged' due to using the general-purpose role has_part that, intuitively, does not belong to any specific subject area. From a logical viewpoint, has_part behaves like a *logical* symbol much like the equality symbol or the symbol $\perp$ for contradiction. This example suggests to generalize the initial version of signature decompositon by adding a set of symbols Δ that do not induce dependencies and do not participate in the decomposition. Formally, a *signature Δ-decomposition* is defined just like a signature decomposition, except that each ontology $\mathcal{T}_i$ is allowed to use symbols from Σ_i *and* Δ. This generalization was first proposed by Ponomaryov (2008). In practice, it may not be easy to determine a suitable Δ. In fact, we do not expect signature decompositions to be a push-button technique, but rather envision an iterative and interactive process of understanding and improving the structure of an ontology, where the designer repeatedly chooses sets Δ and analyzes the impact on the resulting decomposition.

It is important to observe that the definition of a signature decomposition, both with and without the set Δ, depends on the language $\mathcal{L}$ used to formulate the ontologies $\mathcal{T}_1, \dots, \mathcal{T}_n$ that realize the signature decomposition (henceforth called *realizations*). In principle, this is a point of concern as it may not be clear which language $\mathcal{L}$ is appropriate here; for example, when decomposing an ontology $\mathcal{T}$ given in a DL, one might expect more fine-grained decompositions if $\mathcal{L}$ is second-order logic (SO) compared to when $\mathcal{L}$ is again a DL. Therefore, the *first aim of this paper is to study in how far decompositions of DL ontologies depend on the language for the realizations*. Fortunately, it turns out that for many standard DLs, decompositions of TBoxes do not depend on whether one uses SO or the DL for realizations. The main tool for proving this and related results is establishing the *parallel interpolation property*, a type of interpolation that has not yet been investigated in the context of ontologies.

In general, one may expect that there can be many distinct and incomparable signature decompositions of a given ontology $\mathcal{T}$. This is another point of concern because facing a large number of incomparable decompositions is likely to be confusing rather than helpful for an ontology designer. Therefore and since finer decompositions are clearly more informative than coarser ones, one would ideally like to have a *unique* finest decomposition to work with. *Thus, the second aim of this paper is to investigate when unique finest decompositions exist*. Fortunately, we can use parallel interpolation to show that this is the case for many standard DLs.

Syntax	FO	$\mathcal{EL}$	$\mathcal{ALC}$	Short
$\top$	$x = x$	✓	✓	
$\perp$	$\neg(x = x)$		✓	
A	$A(x)$	✓	✓	
$\neg C$	$\neg C(x)$		✓	
$C \sqcap D$	$C(x) \wedge D(x)$	✓	✓	
$\exists r.C$	$\exists y\, (r(x,y) \wedge C(y))$	✓	✓	
$(\leqslant n\ r\ C)$	$\exists^{\geq n} y\, (r(x,y) \wedge C(y))$			$\mathcal{Q}$
$\{a\}$	$x = a$			$\mathcal{O}$
r^-	$r(y,x)$			$\mathcal{I}$
$C \sqsubseteq D$	$\forall x\, (C(x) \rightarrow D(x))$	✓	✓	
$r \sqsubseteq s$	$\forall xy\, (r(x,y) \rightarrow s(x,y))$			$\mathcal{H}$

Figure 1: Standard translation

Finally, we provide *a first analysis of the complexity of some computational problems related to signature decompositions in DL ontologies*. We show that for many expressive DLs, these problems are not harder than standard reasoning. Given that there is a very close connection between signature decompositions on the one hand, and computationally very expensive notions such as conservative extensions and uniform interpolation on the other hand, this result is rather surprising. We also show that in the lightweight description logic DL-Lite, signature decompositions can typically be computed in polynomial time. For the lightweight DL $\mathcal{EL}$, we establish the same result for some restricted, but natural cases.

Many proofs are omitted and can be found in the full version of this paper (Konev et al. 2010).

Preliminaries

Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be countably infinite and mutually disjoint sets of concept names (unary predicates), role names (binary predicates), and individual names. We use $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ as the vocabulary for second-order logic (SO), first-order logic (FO), and a variety of DLs. More precisely, we consider SO (and FO) with equality, the predicates from $\mathsf{N_C} \cup \mathsf{N_R}$ and constants from $\mathsf{N_I}$.[1] Matching this vocabulary, second-order quantification is over set variables and binary relation variables. We use $\mathcal{T} \subseteq$ SO and $\mathcal{T} \subseteq_{fin}$ SO to denote that $\mathcal{T}$ is a set, respectively finite set, of SO-sentences; we write $\mathcal{T} \models \varphi$ if φ is an SO-sentence that is a consequence of $\mathcal{T}$. A set $\mathcal{T} \subseteq$ SO is *satisfiable* iff $\mathcal{T}$ has a model. Two sets $\mathcal{T}_1 \subseteq$ SO and $\mathcal{T}_2 \subseteq$ SO are *equivalent*, in symbols $\mathcal{T}_1 \equiv \mathcal{T}_2$, if they have the same models or, equivalently, if $\mathcal{T}_1 \models \varphi$ for all $\varphi \in \mathcal{T}_2$ and vice versa. We sometimes write $\mathcal{T}_1 \models \mathcal{T}_2$ as shorthand for '$\mathcal{T}_1 \models \varphi$ for all $\varphi \in \mathcal{T}_2$'. The *signature* $\mathsf{sig}(\varphi)$ of an SO-formula is the set of all predicate and constant symbols (except equality) used in φ. This notion is lifted to sets of sentences in the obvious way. A *fragment* of second-order logic is simply a subset $\mathcal{L} \subseteq$ SO.

Description logics can be viewed as fragments of FO. DL *concepts* are formed starting from concept names by induc-

[1]This is only for uniformity with DLs. The results presented in this paper do not depend on the restricted arity.

tively applying concept constructors such as those shown in the upper part of Figure 1. The choice of different constructors gives rise to different DLs. In the figure, we have marked the constructors of the basic DLs $\mathcal{EL}$ and $\mathcal{ALC}$ and assigned to each additional constructor a letter that allows the systematic appellation of extended DLs. The extension $\mathcal{I}$ is with a role constructor for inverse roles, not a concept constructor. When $\mathcal{I}$ is present, inverse roles can be used inside existential restrictions, number restrictions $\mathcal{Q}$ and role hierarchies $\mathcal{H}$. For details, we refer the reader to (Baader et al. 2003).

To simplify notation, we identify models of SO (and, therefore, of FO and DLs) with interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non-empty domain $\Delta^{\mathcal{I}}$ and a function $\cdot^{\mathcal{I}}$ that assigns a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ to each $A \in \mathsf{N_C}$, a relation $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$ to each $r \in \mathsf{N_C}$, and an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ to each $a \in \mathsf{N_I}$. The extension $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of a DL concept C is defined by the standard inductive translation of C into an FO-formula with one free variable x as shown in Figure 1.

A TBox (or *ontology*) is a finite set of concept inclusions (CIs) $C \sqsubseteq D$, where C, D are concepts. An interpretation *satisfies* a CI $C \sqsubseteq D$ (written $\mathcal{I} \models C \sqsubseteq D$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and a TBox $\mathcal{T}$ (written $\mathcal{I} \models \mathcal{T}$) if $\mathcal{I} \models C \sqsubseteq D$ for all $C \sqsubseteq D \in \mathcal{T}$. In the presence of role hierarchies (indicated by the letter $\mathcal{H}$), TBoxes can also include role inclusions $r \sqsubseteq s$ whose semantics can be found in Figure 1. We will typically not distinguish between DL concepts (resp. TBoxes) and their FO translations. In particular, we often regard DL TBoxes as finite sets of FO-sentences (and thus SO-sentences).

Signature Decomposition

We introduce and illustrate the basic notion of this paper and identify some of its fundamental properties.

Definition 1 (Signature Decomposition) *Let $\mathcal{T} \subseteq_{fin}$ SO, $\Delta \subseteq \mathsf{sig}(\mathcal{T})$ and $\mathcal{L}$ a fragment of SO. A partition $\Sigma_1, \dots, \Sigma_n$ of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ is called a* signature Δ-decomposition of $\mathcal{T}$ in $\mathcal{L}$ *if there are $\mathcal{T}_1, \dots, \mathcal{T}_n \subseteq \mathcal{L}$ such that*

- $\mathsf{sig}(\mathcal{T}_i) \subseteq \Sigma_i \cup \Delta$ *for* $1 \leq i \leq n$;
- $\mathcal{T}_1 \cup \dots \cup \mathcal{T}_n \equiv \mathcal{T}$.

In this case, we say that $\mathcal{T}_1, \dots, \mathcal{T}_n$ realize *the signature Δ-decomposition $\Sigma_1, \dots, \Sigma_n$ in $\mathcal{L}$.*

For simplicity, we will often speak only of Δ-decompositions instead of signature Δ-decompositons. When $\Delta = \emptyset$, we simply drop it and speak only of (signature) decompositions. Note that in contrast to Kourousias and Makinson (2007), we consider only finitely axiomatized theories, which suffices for our purposes. Some proofs actually depend on this assumption.

For any $\mathcal{T}$ and Δ, there exists at least one Δ-decomposition, namely the *trivial decomposition* consisting only of the single set $\mathsf{sig}(\mathcal{T}) \setminus \Delta$. We call a partition $\Sigma_1, \dots, \Sigma_n$ *finer* than a partition $\Pi_1, \dots, \Pi_m$ if they are distinct and for every $i \leq m$ there exist $i_1, \dots, i_k \leq n$ such that $\Pi_i = \bigcup_{\ell \leq k} \Sigma_{i_\ell}$.

Example 2 Let $\mathcal{T}$ be the TBox consisting of $\alpha_1 = (\mathsf{Ball} \sqsubseteq \mathsf{Physical\_Object})$, $\alpha_2 = (\mathsf{Table} \sqsubseteq \mathsf{Physical\_Object})$, $\alpha_3 = (\mathsf{Ball} \sqsubseteq \exists \mathsf{has\_colour}.\top)$, $\alpha_4 = (\mathsf{Table} \sqsubseteq \exists \mathsf{has\_colour}.\top)$, $\alpha_5 = (\mathsf{OrangeBall} \sqsubseteq \mathsf{Ball})$.

For any of $\Delta = \emptyset$, $\Delta = \{\mathsf{Physical\_object}\}$ and $\Delta = \{\mathsf{has\_colour}\}$, there are no non-trivial Δ-decompositions of $\mathcal{T}$ because, intuitively, Ball and Table are connected independently via both Phyical_object and has_colour. In many contexts, one would not regard this as a relevant dependency between the two terms. In fact, for $\Delta = \{\mathsf{Physical\_object}, \mathsf{has\_colour}\}$ the finest Δ-decomposition of $\mathcal{T}$ is $\{\mathsf{Ball}, \mathsf{OrangeBall}\}, \{\mathsf{Table}\}$, realized by $\{\alpha_1, \alpha_3, \alpha_5\}$ and $\{\alpha_2, \alpha_4\}$.

One way to extend $\mathcal{T}$ such that Ball and Table are separated already when choosing $\Delta = \{\mathsf{has\_colour}\}$ is to add $\alpha_6 = (\exists \mathsf{has\_colour}.\top \sqsubseteq \mathsf{Physical\_Object})$. In the resulting $\mathcal{T}'$, the axioms α_1, α_2 become redundant and the finest Δ-decomposition is $\{\mathsf{Physical\_Object}\}, \{\mathsf{Ball}, \mathsf{OrangeBall}\}, \{\mathsf{Table}\}$, realized by $\{\alpha_6\}$, $\{\alpha_3, \alpha_5\}$, $\{\alpha_4\}$.

Finally, note that OrangeBall and Ball cannot be separated in a non-trivial way because one would have to extend Δ by at least one of the two concepts.

Signature decompositions that can be obtained by analyzing the syntactic form of axioms are a special case of signature decompositions in the sense of Definition 1. The following example shows how such syntactic decompositions can be computed.

Example 3 (Syntactic decomposition) Let $\mathcal{T} \subseteq_{fin}$ SO and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$. There always exists a (unique) finest Δ-decomposition $\Sigma_1, \dots, \Sigma_n$ that is *realized by subsets $\mathcal{T}_1, \dots, \mathcal{T}_n$ of $\mathcal{T}$*. We denote this Δ-decomposition by $\mathsf{sdeco}_\Delta(\mathcal{T})$ and call it the *syntactic Δ-decomposition of $\mathcal{T}$*. $\mathsf{sdeco}_\Delta(\mathcal{T})$ can be obtained as the partition of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ induced by the smallest equivalence relation on $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ that contains all pairs (σ_1, σ_2) for which there exists $\alpha \in \mathcal{T}$ with $\{\sigma_1, \sigma_2\} \subseteq \mathsf{sig}(\alpha) \setminus \Delta$. In general, $\mathsf{sdeco}_\Delta(\mathcal{T})$ is of course not the finest decomposition possible. Note that $\mathsf{sdeco}_\Delta(\mathcal{T})$ can be computed in poly-time.

We now establish some basic properties of *decompositons in SO*, i.e., decompositions of ontologies based on realizations $\mathcal{T}_1, \dots, \mathcal{T}_n$ that are formulated in SO. As announced in the introduction, decompositions in SO play a special role in this paper as they are easy to work with and turn out to be equivalent to decompositions in many standard DLs. To formulate SO decompositions succinctly, we write $\exists \sigma.\varphi$ to denote $\exists P.\varphi[P/\sigma]$, where either σ is a predicate and P a fresh predicate variable of the same arity as σ, or σ is an individual constant and P a fresh individual variable. Clearly, $\mathsf{sig}(\exists \sigma.\varphi) = \mathsf{sig}(\varphi) \setminus \{\sigma\}$. $\exists \Sigma.\varphi$ is shorthand for $\exists \sigma_1 \cdots \exists \sigma_n.\varphi$ if $\Sigma = \{\sigma_1, \dots, \sigma_n\}$.

Theorem 4 (Characterization) *Let $\mathcal{T} \subseteq_{fin}$ SO and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$. A partition $\Sigma_1, \dots, \Sigma_n$ of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ is a signature Δ-decomposition of $\mathcal{T}$ in SO iff*

$$\{\exists \overline{\overline{\Sigma_1}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi, \cdots, \exists \overline{\overline{\Sigma_n}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi\} \models \mathcal{T} \qquad (*)$$

where $\overline{\overline{\Sigma_i}} := \bigcup_{1 \leq j \leq n, j \neq i} \Sigma_j$.

Proof. "⇒". Assume that the partition $\Sigma_1, \ldots, \Sigma_n$ of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ is a Δ-decomposition of $\mathcal{T}$ in SO realized by $\mathcal{T}_1, \ldots, \mathcal{T}_n$. To show that ($*$) holds, let $\mathcal{I}$ be a model of the left-hand side of ($*$). Then $\mathcal{I}$ is a model of $\mathcal{T}_i$ for $1 \leq i \leq n$: since $\mathcal{I} \models \exists\overline{\overline{\Sigma_i}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi$, there is a model $\mathcal{J}$ of $\mathcal{T}$ that agrees with $\mathcal{I}$ on the interpretation of all symbols from $\Sigma_i \cup \Delta$; since $\mathcal{T} \models \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$, we have $\mathcal{J} \models \mathcal{T}_i$ and due to $\mathsf{sig}(\mathcal{T}_i) \subseteq \Sigma_i \cup \Delta$, it follows that $\mathcal{I} \models \mathcal{T}_i$ as stated. Thus $\mathcal{I} \models \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$ and $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n \models \mathcal{T}$ yields that $\mathcal{I}$ is a model of $\mathcal{T}$, as required.

"⇐" If ($*$) holds, then $\mathcal{T}_i = \{\exists\overline{\overline{\Sigma_i}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi\}$, $1 \leq i \leq n$, clearly realize $\Sigma_1, \ldots, \Sigma_n$. ❑

As a consequence of the proof of Theorem 4, for each decomposition $\Sigma_1, \ldots, \Sigma_n$ in SO, there exists a realization of the canonical (though rather uninformative) form $\mathcal{T}_i = \{\exists\overline{\overline{\Sigma_i}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi\}$, $1 \leq i \leq n$. Clearly, this canonical form relies on second-order quantifiers and does not exist in (fragments of) FO. As a first application of Theorem 4, one can show that there always exists a unique finest Δ-decomposition in SO.

Theorem 5 (Unique Finest Decomposition) *Let $\mathcal{T} \subseteq_{fin}$ SO, $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, and let $\Sigma_1, \ldots, \Sigma_n$ and $\Pi_1, \ldots, \Pi_m$ be Δ-decompositions of $\mathcal{T}$ in SO. Then the partition $\Sigma_i \cap \Pi_j$ for all i, j with $\Sigma_i \cap \Pi_j \neq \emptyset$ of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ is a Δ-decomposition of $\mathcal{T}$ in SO. Thus, there exists a unique finest Δ-decomposition of $\mathcal{T}$ in SO.*

In the following example, we compute the finest Δ-decomposition in SO of concept hierarchies.

Example 6 Let $\mathcal{T}$ be a concept hierarchy, i.e., a finite set of inclusions $A \sqsubseteq B$ between concept names A, B. A realization of the unique finest Δ-decomposition in SO of $\mathcal{T}$ is obtained by first adding to $\mathcal{T}$ all CIs $A \sqsubseteq B$ with $\mathcal{T} \models A \sqsubseteq B$ that contain at most one non-Δ symbol. Then remove from the resulting TBox all $A \sqsubseteq B$ with two non-Δ-symbols for which there exists $D \in \Delta$ with $A \sqsubseteq D, D \sqsubseteq B \in \mathcal{T}$, and denote by $\mathcal{T}'$ the resulting TBox. It can be shown that $\mathsf{sdeco}_\Delta(\mathcal{T}')$ is the unique finest Δ-decomposition of $\mathcal{T}$ in SO which, in this case, is realized using again a concept hierarchy and no second-order quantifiers.

Although there is always a unique finest decomposition in SO, the theories that realize this (finest) decomposition are generally not uniquely determined. To see this, consider the TBox $\mathcal{T} = \{\top \sqsubseteq A \sqcap B_1 \sqcap B_2\}$ and let $\Delta = \{A\}$. Then the unique finest Δ-decomposition $\{B_1\}, \{B_2\}$ is realized by both $\{\top \sqsubseteq A \sqcap B_1\}, \{\top \sqsubseteq B_2\}$ and $\{\top \sqsubseteq B_1\}, \{\top \sqsubseteq A \sqcap B_2\}$. Clearly, there are no two sets in these two realizations that are logically equivalent.

We now present a condition under which realizations are unique, for many fragments of SO. Say that $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{L}$ are Δ-*inseparable w.r.t.* $\mathcal{L}$ if, and only if, $\mathcal{T}_1 \models \varphi$ iff $\mathcal{T}_2 \models \varphi$ for all φ in $\mathcal{L}$ such that $\mathsf{sig}(\varphi) \subseteq \Delta$. Clearly, if Δ contains the signatures $\mathsf{sig}(\mathcal{T}_1)$ and $\mathsf{sig}(\mathcal{T}_2)$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$ iff they are logically equivalent. Otherwise, Δ-inseparability is weaker than logical equivalence and is an extension of the notion of a *conservative extension* (for which, in addition to being Δ-inseparable, it is required that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\Delta = \mathsf{sig}(\mathcal{T}_1)$) that has been used to develop a formal framework for modular ontologies and module extraction (Konev et al. 2009; Lutz and Wolter 2010). Note that for the canonical realization $\mathcal{T}_i = \{\exists\overline{\overline{\Sigma_i}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi\}$, $1 \leq i \leq n$, of Theorem 4 we have that $\mathcal{T}_i, \mathcal{T}_j$ are Δ-inseparable w.r.t. SO for all $1 \leq i, j \leq n$.

Definition 7 (Unique Decomposition Realizations) *Let $\mathcal{L}$ be a fragment of SO. We say that $\mathcal{L}$* has unique decomposition realizations *(UDR) if for all satisfiable $\mathcal{T} \subseteq_{fin} \mathcal{L}$ and all finite $\mathcal{L}$-realizations $\mathcal{T}_1, \ldots, \mathcal{T}_n$ and $\mathcal{T}_1', \ldots, \mathcal{T}_n'$ of a Δ-decomposition of $\mathcal{T}$ such that*

- *$\mathcal{T}_i, \mathcal{T}_j$ are Δ-inseparable w.r.t. $\mathcal{L}$ for $i, j \leq n$ and*
- *$\mathcal{T}_i', \mathcal{T}_j'$ are Δ-inseparable w.r.t. $\mathcal{L}$ for $i, j \leq n$,*

we have $\mathcal{T}_i \equiv \mathcal{T}_i'$ for all $i \leq n$.

UDR has interesting consequences. For example, if $\mathcal{T}_1, \ldots, \mathcal{T}_n$ satisfy the conditions of Definition 7 and $\mathcal{L}$ has UDR, then one can show that $\mathcal{T}$ is a conservative extension of each $\mathcal{T}_i$ (i.e., $\mathcal{T}_i \models \varphi$ iff $\mathcal{T} \models \varphi$ for all φ with $\mathsf{sig}(\varphi) \subseteq \mathsf{sig}(\mathcal{T}_i)$). Thus, realizations satisfy the basic conditions for logic-based ontology modules as proposed and discussed in (Cuenca Grau et al. 2006; 2008; Konev et al. 2009).

Theorem 8 *SO has UDR.*

One can show that the canonical realization provided by Theorem 4 satisfies the conditions of Definition 7 for SO. Therefore, by Theorem 8, all realizations of a given Δ-decomposition that satisfy the conditions of Definition 7 for SO are equivalent to its canonical realization.

In this section, we have seen that decompositions in SO have a variety of desirable properties. The aim of the next section is to investigate in how far these are also enjoyed by decompositions in DLs.

Signature decompositions and parallel interpolation in DLs

By definition, if $\mathcal{L}_1$ is a fragment of $\mathcal{L}_2$, then every Δ-decomposition of some $\mathcal{T}$ in $\mathcal{L}_1$ is a Δ-decomposition of $\mathcal{T}$ in $\mathcal{L}_2$. In particular, every Δ-decomposition of $\mathcal{T}$ in some fragment of SO is a Δ-decomposition of $\mathcal{T}$ in SO. In this section, we show that for many DLs the converse implication holds as well and that, therefore, DLs inherit many of the desirable properties of decompositions in SO.

Definition 9 ($\mathcal{L}$-decompositions = SO-decompositions) *Let $\mathcal{L}$ be a fragment of SO. We say that* $\mathcal{L}$-decompositions coincide with SO-decompositions *if for every $\mathcal{T} \subseteq_{fin} \mathcal{L}$ and every signature $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, the Δ-decompositions of $\mathcal{T}$ in $\mathcal{L}$ coincide with the Δ-decompositions of $\mathcal{T}$ in SO.*

Before we provide methodologies for proving this property for a wide range of DLs, we provide a counterexample showing that $\mathcal{ALCO}$-decompositions do not coincide with SO-decompositions.

Example 10 Let $\Delta = \emptyset$ and $\mathcal{T}$ consist of the $\mathcal{ALCO}$-inclusions

$$\{a\} \sqsubseteq (\exists r.\neg\{a\}) \sqcap (\forall r.\neg\{a\}), \quad \top \sqsubseteq \{b\} \sqcup \{b'\},$$

$$\neg\{a\} \sqsubseteq (\exists r.\{a\}) \sqcap (\forall r.\{a\}), \quad \{a'\} \sqsubseteq \{a'\}.$$

By the CI $\top \sqsubseteq \{b\} \sqcup \{b'\}$, each model of $\mathcal{T}$ has at most two domain elements. Using the two CIs involving a it is, therefore, easy to see that $\mathcal{T}$ axiomatizes the class of two-element interpretations in which b, b' denote distinct elements and r is a symmetric and irreflexive relation that connects the two domain elements. In particular, $\mathcal{T}$ "says nothing" about a and a'. Thus, the finest Δ-decomposition in SO (and FO) of $\mathcal{T}$ is $\{a\}, \{a'\}, \{r\}, \{b, b'\}$. In contrast, one can show that there is no finer Δ-decomposition of $\mathcal{T}$ in $\mathcal{ALCO}$ than $\mathsf{sdeco}_\Delta(\mathcal{T})$ which coincides with $\{a, r\}, \{a'\}, \{b, b'\}$. Another Δ-decomposition of $\mathcal{T}$ in $\mathcal{ALCO}$, which is incompatible with $\mathsf{sdeco}_\Delta(\mathcal{T})$, is given by $\{a', r\}, \{a\}, \{b, b'\}$. It follows that $\mathcal{ALCO}$ TBoxes do not always have a unique finest Δ-decomposition in $\mathcal{ALCO}$.

We now introduce an interpolation property that is not only sufficient to prove that SO-decompositions coincide with $\mathcal{L}$-decompositions, but also implies UDR.

Definition 11 (Parallel Interpolation) *Let $\mathcal{L}$ be a fragment of SO, $(\mathcal{T}_1, \mathcal{T}_2)$ be two sets of SO-sentences, α an SO-sentence with $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$, and Δ a signature. A pair $(\mathcal{T}_1', \mathcal{T}_2')$ with $\mathcal{T}_i' \subseteq \mathcal{L}$ for $i = 1, 2$ is called a* Δ-parallel interpolant *of $(\mathcal{T}_1, \mathcal{T}_2)$ and α in $\mathcal{L}$ if the following conditions hold:*

- $\mathcal{T}_i \models \mathcal{T}_i'$ *for* $i = 1, 2$;
- $\mathsf{sig}(\mathcal{T}_i') \setminus \Delta \subseteq \mathsf{sig}(\mathcal{T}_i) \cap \mathsf{sig}(\alpha)$ *for* $i = 1, 2$;
- $\mathcal{T}_1' \cup \mathcal{T}_2' \models \alpha$.

$\mathcal{L}$ has the parallel interpolation property (PIP) if for all $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{L}$, all $\alpha \in \mathcal{L}$, and all signatures Δ such that

1. $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$,
2. $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$,
3. *$\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$,*

there exists a Δ-parallel interpolant of $(\mathcal{T}_1, \mathcal{T}_2)$ and α in $\mathcal{L}$.

The main reason for studying parallel interpolation is the following result.

Theorem 12 *Let $\mathcal{L}$ be a fragment of SO with the PIP. Then*

1. *$\mathcal{L}$-decompositions coincide with SO-decompositions.*
2. *$\mathcal{L}$ has UDR.*

In particular, every $\mathcal{T} \subseteq_{fin} \mathcal{L}$ has a unique finest Δ-decomposition in $\mathcal{L}$.

Proof. (Sketch for Point 1) Assume that Σ_1, Σ_2 is a Δ-decomposition in SO of $\mathcal{T}$. It follows from Theorem 4 that $\{\exists\Sigma_2. \bigwedge_{\varphi\in\mathcal{T}} \varphi, \exists\Sigma_1. \bigwedge_{\varphi\in\mathcal{T}} \varphi\} \models \mathcal{T}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the subsets of $\mathcal{L}$ obtained from $\mathcal{T}$ by replacing all predicates in Σ_2 and Σ_1, respectively, by fresh predicates. Then $\mathcal{S}_1 \cup \mathcal{S}_2 \models \mathcal{T}$ and the componentwise union of the Δ-parallel interpolants of $(\mathcal{S}_1, \mathcal{S}_2)$ and α in $\mathcal{L}$, $\alpha \in \mathcal{T}$, realizes Σ_1, Σ_2 in $\mathcal{L}$. ❑

The proof shows that an algorithm computing Δ-parallel interpolants in $\mathcal{L}$ can be directly employed to construct a realization in $\mathcal{L}$ of a given Δ-decomposition. As the focus of this paper is on signature decompositions rather than realizations, we concentrate on proving the PIP and leave the computation of Δ-parallel interpolants for future work.

In FO, it is easy to prove the equivalence of the PIP and the standard Craig interpolation property (Parikh 1999; Kourousias and Makinson 2007). Unfortunately, this is not the case for DLs because the proof uses the fact that FO-sentences are closed under Boolean operations and this typically does not hold for DLs (e.g., there does not exist a TBox $\mathcal{T}$ in $\mathcal{ALC}$ that is equivalent to $\neg(\top \sqsubseteq A)$). This also implies that recent results on the existence and computation of Craig interpolants in DL using tableaux are not directly applicable (Seylan, Franconi, and de Bruijn 2009). Nevertheless, it turns our that many DLs have the PIP:

Theorem 13 *The following DLs have the PIP: $\mathcal{EL}$, $\mathcal{ELH}$, $\mathcal{ALC}$, $\mathcal{ALCI}$, $\mathcal{ALCQ}$, $\mathcal{ALCQI}$.*

With the exception of $\mathcal{ELH}$, for which a proof is given in the full version of this paper, Theorem 13 is proved by employing known results regarding the interpolation and Robinson joint consistency properties of DLs. Namely, let $\mathcal{L}$ be a set of sentences in FO. We say that $\mathcal{L}$ has the *Robinson Joint Consistency Property (RJCP)* if the following holds for all $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{L}$ and all signatures Δ: if $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$, then

$$\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha \quad \Leftrightarrow \quad \mathcal{T}_1 \models \alpha$$

for all sentences α in $\mathcal{L}$ with $\mathsf{sig}(\alpha) \subseteq \mathsf{sig}(\mathcal{T}_1)$. We say that $\mathcal{L}$ has the *Boolean Craig Interpolation Property (BCIP)* if for all $\mathcal{T} \subseteq \mathcal{L}$ and all Boolean combinations φ of $\mathcal{L}$-sentences the following holds: if $\mathcal{T} \models \varphi$, then there exists a Boolean combination ψ of $\mathcal{L}$-sentences with $\mathsf{sig}(\psi) \subseteq \mathsf{sig}(\mathcal{T}) \cap \mathsf{sig}(\varphi)$ such that $\mathcal{T} \models \psi$ and $\psi \models \varphi$. Finally, we say that $\mathcal{L}$ has the *disjoint union property* if the following holds for all $\mathcal{T} \subseteq \mathcal{L}$: for all families $\mathcal{I}_i$, $i \in I$, of interpretations the following conditions are equivalent:

- all $\mathcal{I}_i$, $i \in I$, are models of $\mathcal{T}$;
- the disjoint union of all $\mathcal{I}_i$, $i \in I$, is a model of $\mathcal{T}$.

Note that $\mathcal{EL}$, $\mathcal{ELH}$, $\mathcal{ALC}$, $\mathcal{ALCQI}$ and all standard dialects of DL-Lite have the disjoint union property. Examples of DLs without the disjoint union property are DLs with nominals or the universal role. Now one can prove the following equivalences.

Theorem 14 *Let $\mathcal{L}$ be a fragment of FO with the disjoint union property. Then the following conditions are equivalent:*

- *$\mathcal{L}$ has the PIP;*
- *$\mathcal{L}$ has RJCP;*
- *$\mathcal{L}$ has the BCIP.*

We come to the proof of Theorem 13: the PIP of $\mathcal{EL}$ follows from Theorem 34 and its RJCP proved in (Lutz and Wolter 2010). The PIP of $\mathcal{ALC}$, $\mathcal{ALCQ}$, $\mathcal{ALCI}$, and $\mathcal{ALCQI}$ follows from Theorem 34 and their BCIP proved in (Konev et al. 2009). It remains to apply Theorem 34.

It is interesting to observe that the addition of role inclusions to $\mathcal{EL}$ preserves the PIP. This is true for DL-Lite (see the analysis below) as well, but expressive DLs with role inclusions typically do not have the PIP:

Example 15 $\mathcal{ALCH}$ does not have the PIP. Let $\Delta = \{r_1, r_2\}$, $\alpha = \forall r_1.A \sqsubseteq \exists r_2.A$, $\mathcal{T}_1 = \{\top \sqsubseteq \exists r_1.\top \sqcap \exists r_2.\top\}$, and $\mathcal{T}_2 = \{s \sqsubseteq r_1, s \sqsubseteq r_2, \top \sqsubseteq \exists s.\top\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$ but there does not exist a Δ-parallel interpolant of $(\mathcal{T}_1, \mathcal{T}_2)$ and α in $\mathcal{ALCH}$. We note that it remains an open problem whether $\mathcal{ALCH}$-decompositions coincide with SO-decomposition.

We now show how the PIP can be restored for expressive DLs with role inclusions and/or nominals by including into Δ all role and individual names. To obtain the PIP in the presence of nominals we take, in addition, the @-operator from hybrid logic (Areces and ten Cate 2006) (an alternative approach to restoring the PIP is to admit Boolean TBoxes or, equivalently, the universal role). Given a DL $\mathcal{L}$, we denote by $\mathcal{L}$@ the DL obtained from $\mathcal{L}$ by adding the @-operator as a new concept constructor: if a is an individual name and C an $\mathcal{L}$@-concept, then $@_a C$ is an $\mathcal{L}$@ concept. In every interpretation $\mathcal{I}$, $(@_a C)^{\mathcal{I}} = \Delta^{\mathcal{I}}$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and $(@_a C)^{\mathcal{I}} = \emptyset$ otherwise. The following theorem can now be proved by extending results and techniques introduced in (ten Cate 2005; ten Cate et al. 2006).

Theorem 16 *Assume $\mathcal{L} \in \{\mathcal{ALCH}, \mathcal{ALCHI}, \mathcal{ALCO}@, \mathcal{ALCHO}@, \mathcal{ALCHIO}@\}$. Then Δ-parallel interpolants exist in $\mathcal{L}$ for every $(\mathcal{T}_1, \mathcal{T}_2)$ in $\mathcal{L}$ and $\mathcal{L}$-inclusion α such that 1.–3. from Definition 11 hold and Δ contains all role and individual names in $\mathcal{T}_1, \mathcal{T}_2, \alpha$.*

In particular, for every $\mathcal{T}$ in $\mathcal{L}$ and Δ containing all role and individual names in $\mathcal{T}$, Δ-decompositions of $\mathcal{T}$ in SO coincide with Δ-decompositions of $\mathcal{T}$ in $\mathcal{L}$.

Computing decompositions in expressive DLs

We now exploit the results of the previous two sections to analyze the computational complexity of the problem of computing, given $\mathcal{T} \subseteq_{fin} \mathcal{L}$ and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, the finest Δ-decomposition of $\mathcal{T}$ in $\mathcal{L}$. We confine ourselves to languages $\mathcal{L}$ in which SO-decompositions coincide with $\mathcal{L}$-decompositions and, therefore, can assume that unique finest decompositions always exist and coincide with the finest Δ-decomposition in SO. In this section, we prove tight complexity bounds for a range of expressive DLs; in the next section, we consider lightweight DLs.

It will be convenient to reformulate the problem of computing the finest Δ-decomposition as a decision problem. Say that a signature Σ (concept C, CI α) is *Δ-decomposable w.r.t. a TBox $\mathcal{T}$* iff there exists a Δ-decomposition $\Sigma_1, \ldots, \Sigma_n$ of $\mathcal{T}$ such that $\Sigma \not\subseteq \Sigma_i \cup \Delta$ ($\mathsf{sig}(C) \not\subseteq \Sigma_i \cup \Delta$, $\mathsf{sig}(\alpha) \not\subseteq \Sigma_i \cup \Delta$) for all $i \leq n$. *Deciding Δ-decomposability in $\mathcal{L}$* means, given a TBox $\mathcal{T}$ in $\mathcal{L}$, $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, and $\sigma_1, \sigma_2 \in \mathsf{sig}(\mathcal{T})$, to check whether σ_1 and σ_2 are Δ-decomposable w.r.t. $\mathcal{T}$. Δ-decomposability may be viewed as the decision problem associated with computing the finest Δ-decomposition of $\mathcal{T}$: it is not difficult to see that the finest Δ-decomposition of $\mathcal{T}$ coincides with the partition of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ induced by the equivalence relation $\sim$ defined by setting $\sigma_1 \sim \sigma_2$ iff $\{\sigma_1, \sigma_2\}$ are Δ-indecomposable w.r.t. $\mathcal{T}$.

Theorem 17 (Complexity of Δ-decomposability) *In $\mathcal{ALC}$, $\mathcal{ALCI}$, $\mathcal{ALCQ}$, or $\mathcal{ALCQI}$, Δ-decomposability is* ExpTime*-complete.*

Proof. We start with the upper bound. Assume a TBox $\mathcal{T}$ in $\mathcal{L}$, a signature $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, and $\sigma_1, \sigma_2 \in \mathsf{sig}(\mathcal{T}) \setminus \Delta$ are given. Enumerate all (exponentially many) partitions Σ_1, Σ_2 of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$ such that $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. Then σ_1, σ_2 are Δ-decomposable w.r.t. $\mathcal{T}$ if, and only if, at least one these partitions is a Δ-decomposition of $\mathcal{T}$. It is thus sufficient to show that the latter problem can be decided in ExpTime. Assume Σ_1, Σ_2 is given. By Theorem 4, Σ_1, Σ_2 is a Δ-decomposition of $\mathcal{T}$ in SO (and, therefore, by the PIP, in $\mathcal{L}$) if, and only if,

$$\{\exists \Sigma_2. \bigwedge_{C \sqsubseteq D \in \mathcal{T}} C \sqsubseteq D, \exists \Sigma_1. \bigwedge_{C \sqsubseteq D \in \mathcal{T}} C \sqsubseteq D\} \models \mathcal{T}.$$

By introducing fresh predicates for the existentially quantified variables, this condition can be checked using standard subsumption checking w.r.t. $\mathcal{L}$-TBoxes, thus in ExpTime (Baader et al. 2003). For the ExpTime-lower bound, observe that a TBox $\mathcal{T}$ is unsatisfiable iff A, B are Δ-decomposable w.r.t. $\mathcal{T} \cup \{A \sqsubseteq B\}$, where A, B are concept names that do not occur in $\mathcal{T}$. Checking unsatisfiability of TBoxes in $\mathcal{L}$ is ExpTime-hard (Baader et al. 2003). ❑

Clearly, this proof does not provide a practical method for computing finest decompositions. For expressive DLs we leave as future work. Theorem 17 can be generalized in various directions. In the proof, we did not use any specific properties of $\mathcal{L}$, except that SO-decompositions coincide with $\mathcal{L}$-decompositions. Thus the same proof can be used to show that for any such language $\mathcal{L}$ in which subsumption is at least ExpTime-hard, checking Δ-decomposability of two symbols is of the same complexity as subsumption. Together with Theorem 16, we also obtain the following result.

Theorem 18 *In $\mathcal{ALCH}$, $\mathcal{ALCHI}$, $\mathcal{ALCO}@$, $\mathcal{ALCHO}@$, and $\mathcal{ALCHIO}@$, Δ-decomposability with Δ containing all role and invidual names from the input TBox is* ExpTime*-complete.*

For languages $\mathcal{L}$ in which reasoning is strictly less complex than ExpTime, the proof does not necessarily work because the enumeration step for the signature partitions requires exponential time already. In particular, we cannot use the proof to establish tractability of Δ-decomposability for DLs such as DL-Lite and $\mathcal{EL}$ in which subsumption is tractable.

Decomposition in DL-Lite

Our aim in this section is to establish the PIP and prove tractability of computing the finest Δ-decomposition for members of the DL-Lite family of description logics (Calvanese et al. 2009). We start by investigating the basic language DL-Lite$_{core}$ and then move via DL-Lite$_{horn}$ and full DL-Lite$_{horn}$ to DL-Lite$^{\mathcal{H}}_{horn}$, the extension of DL-Lite$_{horn}$ with role hierarchies. Using the techniques introduced in this section, it is rather straightforward to extend the results presented here to other DL-Lite dialects such as DL-Lite$_{\mathcal{R}}$, DL-Lite$_{\mathcal{F}}$, and DL-Lite$^{\mathcal{N}}_{horn}$ (Calvanese et al. 2006; Artale et al. 2009). The algorithms in this and the subsequent section work by first converting the input TBox $\mathcal{T}$ into an equivalent TBox $\mathcal{T}'$ in which every CI is Δ-indecomposable w.r.t. $\mathcal{T}$. It is not hard to show that, then, $\mathsf{sdeco}_\Delta(\mathcal{T}')$ coincides with the finest Δ-decomposition of $\mathcal{T}'$, and thus of $\mathcal{T}$. In contrast to the "non-constructive" second-order approach underlying the proof of Theorem 17, this also allows to compute a realization $\mathcal{T}_1, \ldots, \mathcal{T}_n$ formulated in the same language as the input TBox $\mathcal{T}$.

Recall that *basic DL-Lite concepts* B are defined as

$$B \ ::= \ \top \ | \ \bot \ | \ A \ | \ \exists r \ | \exists r^-,$$

where A ranges over $\mathsf{N_C}$ and r over $\mathsf{N_R}$. DL-Lite$_{core}$-inclusions take the form $B_1 \sqsubseteq B_2$ and $B_1 \sqsubseteq \neg B_2$, where B_1, B_2 are basic DL-Lite concepts. A DL-Lite$_{core}$-TBox is a finite set of DL-Lite$_{core}$-inclusions.

Theorem 19 *DL-Lite$_{core}$ has the PIP. For DL-Lite$_{core}$-TBoxes $\mathcal{T}$ and signatures Δ, one can compute in polynomial time a realization in DL-Lite$_{core}$ of the finest Δ-decomposition $\mathcal{T}$.*

Proof. The algorithm is rather straightforward and almost identical to the algorithm for concept hierarchies in Example 6. First add to $\mathcal{T}$ all DL-Lite$_{core}$ CIs $B_1 \sqsubseteq B_2$ with $\mathcal{T} \models B_1 \sqsubseteq B_2$ and containing not more than one non-Δ-symbol. Now remove from the resulting TBox all $B_1 \sqsubseteq B_2$ containing two non-Δ-symbols for which there exists a concept D that is either a basic DL-Lite concept or its negation and such that $\mathsf{sig}(D) \subseteq \Delta$ and $\mathcal{T} \models B_1 \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq B_2$. For the resulting TBox $\mathcal{T}'$, one can show that $\mathsf{sdeco}_\Delta(\mathcal{T}')$ coincides with the finest Δ-decomposition of $\mathcal{T}$. The correctness of this algorithm and the PIP are proved in the full version, but can also be derived from results for more expressive DL-Lite dialects given below. ❑

The construction above can easily be generalized to DL-Lite dialects admitting no conjunctions on the left-hand side of CIs such as DL-Lite$_{\mathcal{R}}$, DL-Lite$_{\mathcal{F}}$, and the dialect underpinning OWL2-QL.

The construction of realizations of finest Δ-decompositions becomes more involved if axioms with conjunctions on the left hand side of CIs are admitted. To illustrate our approach, we provide an example.

Example 20 Let $\Delta = \{D_1, D_2\}$ and

$$\mathcal{T} = \{A_1 \sqcap A_2 \sqsubseteq B, A_1 \sqsubseteq D_1, A_2 \sqsubseteq D_2, D_1 \sqcap D_2 \sqsubseteq A_1\}.$$

In the spirit of the proof of Theorem 19, let us try to replace CIs in $\mathcal{T}$ to make $\mathsf{sdeco}_\Delta(\mathcal{T})$ as fine-grained as possible. Since no CI except $\alpha_0 = (A_1 \sqcap A_2 \sqsubseteq B)$ contains more than one non-Δ-symbol, all CIs distinct from α_0 are Δ-indecomposable w.r.t. $\mathcal{T}$ and replacing them is of no help. So the only CI we attempt to replace is α_0. Intuitively, α_0 is Δ-decomposable w.r.t. $\mathcal{T}$ because $A_1 \sqcap A_2$ is equivalent to a concept not using A_1 in $\mathcal{T} \setminus \{\alpha_0\}$. More precisely,

$$\mathcal{T} \setminus \{\alpha_0\} \models (A_1 \sqcap A_2) \equiv (D_1 \sqcap A_2).$$

Thus we can replace in $\mathcal{T}$ the CI α_0 by $D_1 \sqcap A_2 \sqsubseteq B$. The resulting TBox realizes the partition $\{A_2, B\}, \{A_1\}$ which can be shown to be the finest Δ-decomposition of $\mathcal{T}$. Note that we could have used $D_1 \sqcap D_2 \sqcap A_2$ instead of $D_1 \sqcap A_2$.

Example 20 suggests to extend the algorithm in the proof of Theorem 19 as follows: for each CI $C_0 \sqsubseteq B_0$ in a TBox $\mathcal{T}$ under consideration, we check whether C_0 can be replaced by a concept C_0' with $\mathsf{sig}(C_0') \setminus \Delta \subsetneq \mathsf{sig}(C_0) \setminus \Delta$ such that

$$\mathcal{T} \equiv (\mathcal{T} \setminus \{C_0 \sqsubseteq B_0\}) \cup \{C_0' \sqsubseteq B_0\}.$$

When searching for such a C_0', it turns out to be sufficient to consider concepts C_0' that are equivalent to C_0 w.r.t. $\mathcal{T} \setminus \{C_0 \sqsubseteq B_0\}$. In other words, it is sufficient to search for an *explicit definition*

$$C_0 \equiv C_0'$$

of C_0 that follows from $\mathcal{T} \setminus \{C_0 \sqsubseteq B_0\}$ and in which C_0' is a concept using less non-Δ-symbols than C_0. If one adopts this approach, it remains to find a polytime algorithm searching for explicit definitions of a concept C_0 within a signature Σ. In the case of DL-Lite, one can employ the following greedy algorithm: for a finite signature Σ, let $\mathsf{Cons}_{\mathcal{T},\Sigma}(C_0)$ consist of all basic DL-Lite concepts D with $\mathsf{sig}(D) \subseteq \Sigma$ such that $\mathcal{T} \models C_0 \sqsubseteq D$. This set is finite (in fact, of linear size in the size of Σ) because there are only linearly many basic DL-Lite concepts over any finite signature. It can also be computed in polynomial time. Thus, we can form the conjunction over all concepts in $\mathsf{Cons}_{\mathcal{T},\Sigma}(C_0)$, which, for simplicity, we denote by $\mathsf{Cons}_{\mathcal{T},\Sigma}(C_0)$ as well. In Example 20, one obtains

$$\mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},\{D_1,D_2,A_2\}}(A_1 \sqcap A_2) = D_1 \sqcap D_2 \sqcap A_2.$$

By definition, $\mathsf{Cons}_{\mathcal{T},\Sigma}(C_0)$ is the *most specific Σ-concept* subsuming C_0 w.r.t. $\mathcal{T}$. Thus, we obtain that there exists an explicit definition C_0' of C_0 w.r.t. $\mathcal{T} \setminus \{C_0 \sqsubseteq B_0\}$ using symbols in Σ only if, and only if,

$$\mathcal{T} \setminus \{C_0 \sqsubseteq B_0\} \models \mathsf{Cons}_{\mathcal{T}\setminus\{C_0 \sqsubseteq B_0\},\Sigma}(C_0) \sqsubseteq C_0,$$

and, if this happens to be the case, then $\mathsf{Cons}_{\mathcal{T}\setminus\{C_0 \sqsubseteq B_0\},\Sigma}(C_0)$ is such a definition. Finally, to test whether there is some Σ containing less non-Δ-symbols than $\mathsf{sig}(C_0)$ with this property, one can go through all

Input: Propositional DL-Lite$_{\text{horn}}$ TBox $\mathcal{T}$ and signature $\Delta \subseteq \mathsf{sig}(\mathcal{T})$.

Apply exhaustively the following transformation rule to each $\alpha = C \sqsubseteq B \in \mathcal{T}$ such that $|\mathsf{sig}(\alpha) \setminus \Delta| \geq 2$.

1. **If** $\mathcal{T} \setminus \{\alpha\} \models \alpha$
2. **Then**
3. $\quad \mathcal{T} := \mathcal{T} \setminus \{\alpha\}$.
4. **Else**
5. $\quad$ **If** $\mathsf{sig}(C) \not\subseteq \Delta$, $\mathsf{sig}(B) \not\subseteq \Delta$, and $\mathcal{T} \models \mathsf{Cons}_{\mathcal{T},\Delta}(C) \sqsubseteq B$
6. $\quad$ **Then**
7. $\quad\quad \mathcal{T} := (\mathcal{T} \setminus \{\alpha\})$;
8. $\quad\quad \mathcal{T} := \mathcal{T} \cup \{\mathsf{Cons}_{\mathcal{T},\Delta}(C) \sqsubseteq B\} \cup \bigcup_{B' \in \mathsf{Cons}_{\mathcal{T},\Delta}(C)} \{C \sqsubseteq B'\}$
9. $\quad$ **If** for some $X \in \mathsf{sig}(C) \setminus \Delta$
10. $\quad\quad \mathcal{T} \setminus \{\alpha\} \models \mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},(\mathsf{sig}(C)\setminus\{X\})\cup\Delta}(C) \sqsubseteq C$
11. $\quad$ **Then**
12. $\quad\quad \mathcal{T} := (\mathcal{T} \setminus \{\alpha\}) \cup \{\mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},(\mathsf{sig}(C)\setminus\{X\})\cup\Delta}(C) \sqsubseteq B\}$

Figure 2: Procedure $\mathsf{Rewrite}_{\text{PropDL-Lite}_{\text{horn}}}$

$\Sigma := (\Delta \cup \mathsf{sig}(C_0)) \setminus \{X\}$ for $X \in \mathsf{sig}(C_0) \setminus \Delta$. Since the finest decomposition is unique, the order in which we go through such Σ's does not matter.

We now present the algorithm implementing this approach in detail. Recall that a DL-Lite$_{horn}$-inclusion takes the form $B_1 \sqcap \cdots \sqcap B_m \sqsubseteq B$, where the $B_1, \ldots, B_m$ and B are basic DL-Lite concepts. We first consider *propositional DL-Lite$_{horn}$*, i.e., DL-Lite$_{horn}$-inclusions and TBoxes not containing any roles. Of course propositional DL-lite$_{horn}$ is nothing else but propositional Horn-logic. We first observe that DL-Lite$_{\text{horn}}$ and propositional DL-Lite$_{\text{horn}}$ have the PIP; so it does not make any difference whether we consider signature decompositions realized in DL-Lite$_{horn}$ or in SO:

Lemma 21 *DL-Lite$_{horn}$ and propositional DL-Lite$_{horn}$ have the PIP.*

Proof. It is shown in (Kontchakov, Wolter, and Zakharyaschev 2010) that DL-Lite$_{\text{horn}}$ has the RJCP. By Theorem 34, DL-Lite$_{\text{horn}}$ has the PIP. The same proof works for propositional DL-Lite$_{\text{horn}}$. ❏

Theorem 22 *For any propositional DL-Lite$_{horn}$ TBox $\mathcal{T}$, the algorithm in Figure 2 runs in poly-time and outputs an equivalent TBox $\mathcal{T}'$ in which every CI is Δ-indecomposable w.r.t. $\mathcal{T}$. Thus,* $\mathsf{sdeco}_\Delta(\mathcal{T}')$ *coincides with the finest Δ-decomposition of $\mathcal{T}$.*

Proof. We provide a sketch of the correctness proof; a detailed proof can be found in the full version. Denote by $\mathcal{T}$ the output of the algorithm in Figure 2. It can be verified that this TBox is equivalent to the original TBox. Moreover it has the following properties:

(Red) For every $\alpha \in \mathcal{T}$ with $|\mathsf{sig}(\alpha) \setminus \Delta| \geq 2$, we have $\mathcal{T} \setminus \{\alpha\} \not\models \alpha$;

(Def) If for some $\alpha = (C \sqsubseteq B) \in \mathcal{T}$ and $\Sigma \subseteq \mathsf{sig}(C) \setminus \Delta$ we have
$$\mathcal{T} \setminus \{\alpha\} \models \mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},\Sigma\cup\Delta}(C) \sqsubseteq C,$$
then $\Sigma = \mathsf{sig}(C) \setminus \Delta$.

(Int) For any $C \sqsubseteq B \in \mathcal{T}$ such that $\mathsf{sig}(C) \not\subseteq \Delta$ and $\mathsf{sig}(B) \not\subseteq \Delta$, we have
$$\mathcal{T} \not\models \mathsf{Cons}_{\mathcal{T},\Delta}(C) \sqsubseteq B.$$

Thus, it is sufficient to prove the following

Claim. If a TBox $\mathcal{T}$ has properties (Red), (Def), and (Int), then every CI in $\mathcal{T}$ is Δ-indecomposable w.r.t. $\mathcal{T}$.

Proof of Claim. Suppose that some $\alpha = C \sqsubseteq B \in \mathcal{T}$ is Δ-decomposable w.r.t. $\mathcal{T}$. Then either

(a) there exists a signature Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$ such that $\mathsf{sig}(C) \cap \Sigma_i \neq \emptyset$ for $i = 1, 2$ or

(b) there exists a signature Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$ such that $\mathsf{sig}(C) \cap \Sigma_2 \neq \emptyset$, $\mathsf{sig}(C) \subseteq \Sigma_2 \cup \Delta$, and $\mathsf{sig}(B) \subseteq \Sigma_1$.

We show that, in both cases, a contradiction can be derived. We use the following notation for renaming symbols within concepts, CIs, and TBoxes. Let D be a concept. By D_{Σ_1} we denote the concept obtained from D by replacing every occurrence of a symbol $x \in \Sigma_2$ with a fresh symbol x'. By D_{Σ_2} we denote the concept obtained from D by replacing every occurrence of a symbol $x \in \Sigma_1$ with a fresh symbol $x^\dagger$. The CIs α_{Σ_1}, α_{Σ_2} and TBoxes $\mathcal{T}_{\Sigma_1}$, $\mathcal{T}_{\Sigma_2}$ are defined in the same way. Recall from Theorem 4 that Σ_1, Σ_2 is a Δ-decomposition of $\mathcal{T}$ if, and only if,
$$\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2} \models \alpha$$
for all $\alpha \in \mathcal{T}$.

Consider now Case (a). By (Red), we have $\mathcal{T} \setminus \{\alpha\} \not\models \alpha$. Therefore,
$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \not\models \alpha. \tag{1}$$
On the other hand,
$$\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2} \models \alpha,$$
since Σ_1, Σ_2 is a Δ-decomposition of $\mathcal{T}$. Thus, there exists $i \in \{1, 2\}$ such that
$$\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2} \models C \sqsubseteq C_{\Sigma_i} \tag{2}$$
because otherwise, by (1) we would find a (propositional) model $\mathcal{I}$

- satisfying $(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2}$ and C;
- and refuting C_{Σ_1}, C_{Σ_2}, and B.

For such an $\mathcal{I}$ we would have $\mathcal{I} \models \alpha_{\Sigma_1}$ and $\mathcal{I} \models \alpha_{\Sigma_2}$ and, therefore, $\mathcal{I} \models \mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2}$ but $\mathcal{I} \not\models C \sqsubseteq B$, which is a contradiction.

Now one can show (the proof is non-trivial and given in the full paper) that (2) implies
$$\mathcal{T} \setminus \{\alpha\} \models \mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},\Delta\cup(\Sigma_i\cap\mathsf{sig}(C)))}(C) \sqsubseteq C,$$
which contradicts (Def).

In Case (b), one can show (the proof is non-trivial) that
$$\mathcal{T} \models \mathsf{Cons}_{\mathcal{T},\Delta}(C) \sqsubseteq B,$$
which contradicts (Int). ❏

Input: DL-Lite$_{\text{horn}}$ TBox $\mathcal{T}$ and signature $\Delta \subseteq \mathsf{sig}(\mathcal{T})$.
1. **Let** $\mathcal{T}_{\text{Aux}} := \{\exists r \sqsubseteq \bot \mid \mathcal{T} \models \exists r \sqsubseteq \bot\}$
2. **Let** $\mathcal{T}^P_{\text{Res}} := \mathsf{Rewrite}_{\text{PropDL-Lite}_{\text{horn}}}(\mathcal{T}^P \cup \mathcal{T}^P_{\text{Aux}}, \Delta^P)$
3. Let $\mathcal{T}'$ be the result of replacing in $\mathcal{T}^P_{\text{Res}}$ expressions of the form $P_{\exists r}$, for $r \in \mathsf{N_R}$, with $\exists r$ and $P_{\exists r^-}$ with $\exists r^-$.
4. **Return** $\mathcal{T}'$

Figure 3: Procedure $\mathsf{Rewrite}_{\text{DL-Lite}_{\text{horn}}}$

We now consider DL-Lite$_{\text{horn}}$. The following lemma shows that reasoning in DL-Lite$_{\text{horn}}$ can be reduced to reasoning in propositional DL-Lite$_{\text{horn}}$. Its proof is similar to the ones of results in (Artale et al. 2009) relating DL-Lite dialects and fragments of first-order logic.

Given a DL-Lite$_{\text{horn}}$ concept C (CI $C \sqsubseteq B$ or TBox $\mathcal{T}$, respectively), we consider a propositional DL-Lite$_{\text{horn}}$ concept C^P (propositional CI $C^P \sqsubseteq B^P$ or propositional TBox $\mathcal{T}^P$) obtained by replacing every occurrence of an expression of the form $\exists r$ (resp. $\exists r^-$) with its *surrogate*, a fresh concept name $P_{\exists r}$ (resp. $P_{\exists r^-}$). We assume that surrogates do not occur in the given DL-Lite$_{\text{horn}}$ concept (CI, TBox, respectively). Let Σ be a signature. We define its propositional counterpart as

$$\Sigma^P = \{A \mid A \in \Sigma, A \in \mathsf{N_C}\} \cup \{P_{\exists r}, P_{\exists r^-} \mid r \in \Sigma, r \in \mathsf{N_R}\}.$$

The following is readily checked.

Lemma 23 *Let $C \sqsubseteq B$ be a DL-Lite$_{horn}$ CI, and $\mathcal{T}$ a satisfiable DL-Lite$_{horn}$ TBox such that for all roles r, if $\mathcal{T} \models \exists r \sqsubseteq \bot$, then $\exists r \sqsubseteq \bot \in \mathcal{T}$. Then $\mathcal{T} \models C \sqsubseteq B$ if, and only if, $\mathcal{T}^P \models C^P \sqsubseteq B^P$.*

Using Lemma 23 one can now prove the correctness of the algorithm given in Figure 3.

Theorem 24 *The algorithm $\mathsf{Rewrite}_{\text{DL-Lite}_{horn}}$ given in Fig. 3 transforms a given DL-Lite$_{horn}$ TBox into an equivalent DL-Lite$_{horn}$ TBox in which every CI is Δ-indecomposable.*

Finally, we consider DL-Lite$^{\mathcal{H}}_{\text{horn}}$, the extension of DL-Lite$_{\text{horn}}$ with role inclusions $r \sqsubseteq s$. This time, we employ a reduction of DL-Lite$^{\mathcal{H}}_{\text{horn}}$ to DL-Lite$_{\text{horn}}$.

Lemma 25 *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{horn}$ TBox and Δ a signature. Let $\mathcal{T}^0$ be the set of CIs in $\mathcal{T}$ and set*

$$\mathcal{T}' = \mathcal{T}^0 \cup \{\exists r \sqsubseteq \exists s, \exists r^- \sqsubseteq \exists s^- \mid r \sqsubseteq s \in \mathcal{T}\}.$$

Then $\mathcal{T} \models \alpha$ if, and only if, $\mathcal{T}' \models \alpha$ for all CIs α in DL-Lite$_{horn}$.

Using this reduction, one can show that DL-Lite$^{\mathcal{H}}_{\text{horn}}$ has the PIP and one can prove the correctness of the algorithm given in Figure 4.

Theorem 26 *The algorithm $\mathsf{Rewrite}_{\text{DL-Lite}^{\mathcal{H}}_{horn}}$ given in Fig. 4 transforms a given DL-Lite$^{\mathcal{H}}_{horn}$ TBox into an equivalent DL-Lite$^{\mathcal{H}}_{horn}$ TBox in which every inclusion is Δ-indecomposable.*

Input: DL-Lite$^{\mathcal{H}}_{\text{horn}}$ TBox $\mathcal{T}$ and signature $\Delta \subseteq \mathsf{sig}(\mathcal{T})$.
1. **Let** $\mathcal{T}_C$ be the the set of CI in $\mathcal{T}$
2. **Let** $\mathcal{T}_R$ be the the set of RI in $\mathcal{T}$
3. $\mathcal{T}_C := \mathcal{T}_C \cup \{\exists r \sqsubseteq \exists s \mid \mathcal{T} \models \exists r \sqsubseteq \exists s\} \cup \{\exists r \sqsubseteq \bot \mid \mathcal{T} \models \exists r \sqsubseteq \bot\}$
4. $\mathcal{T}_C := \mathsf{Rewrite}_{\text{DL-Lite}_{\text{horn}}}(\mathcal{T}_C, \Delta)$
5. **For all** $r \sqsubseteq s \in \mathcal{T}_R$ **do**
6. **If** $\mathcal{T}_C \models \exists r \sqsubseteq \bot$
7. **Then** $\mathcal{T}_R := \mathcal{T}_R \setminus \{r \sqsubseteq s\}$
8. **Else if** $\mathcal{T}_R \models r \sqsubseteq t$ and $\mathcal{T}_R \models t \sqsubseteq s$ for some $t \in \Delta$
9. **Then**
10. $\mathcal{T}_R := (\mathcal{T}_R \setminus \{r \sqsubseteq s\}) \cup \{r \sqsubseteq t\} \cup \{t \sqsubseteq s\}$
11. **Return** $(\mathcal{T}_C \cup \mathcal{T}_R)$

Figure 4: Procedure $\mathsf{Rewrite}_{\text{DL-Lite}^{\mathcal{H}}_{\text{horn}}}$

Decomposition in $\mathcal{EL}$

We have seen already (Theorem 13) that $\mathcal{EL}$ and $\mathcal{ELH}$ have the PIP. In this section, we focus on computing the finest Δ-decompositions in $\mathcal{EL}$. In contrast to DL-Lite, we have partial results only. Call an $\mathcal{EL}$-TBox $\mathcal{T}$ *role-acyclic* if there does not exist an $\mathcal{EL}$-concept C and role names $r_1, \dots, r_n$ with $n \geq 1$ such that $\mathcal{T} \models C \sqsubseteq \exists r_1.\cdots\exists r_n.C$ Note that acyclic terminologies such as SNOMED CT satisfy this condition.

Theorem 27 *Let*

1. *$\Delta = \emptyset$ and $\mathcal{T}$ be an arbitrary $\mathcal{EL}$-TBox; or*
2. *Δ arbitrary and $\mathcal{T}$ be a role-acyclic TBox.*

Then the finest Δ-decomposition of $\mathcal{T}$ can be computed in polynomial time.

It remains an open problem whether this results holds for arbitrary $\mathcal{EL}$-TBoxes. We nowgive a sketch of the main ideas behind the proof for Point 2. First, using results from (Lutz and Wolter 2010), one can transform any given $\mathcal{EL}$-TBox $\mathcal{T}_0$ and signature Δ_0 into a new TBox $\mathcal{T}$ and signature Δ (which is, modulo fresh definitions $A \equiv C$, equivalent to $\mathcal{T}_0$) such that the finest Δ-decomposition of $\mathcal{T}$ can be transformed in linear time into the finest Δ_0-decomposition of $\mathcal{T}_0$ and such that:

(Dec) if $C \sqsubseteq D \in \mathcal{T}$, then D is Δ-indecomposable w.r.t. $\mathcal{T}$.

The full version describes how $\mathcal{T}$ and Δ can be computed. Given $\mathcal{T}$ and Δ satisfying (Dec), we want to proceed in the same way as for DL-Lite: a CI $\alpha = (C \sqsubseteq D) \in \mathcal{T}$ should be simplified to $C' \sqsubseteq D$ if C' is an explicit definition of C relative to $\mathcal{T} \setminus \{\alpha\}$ using less non-Δ-symbols than C. This simplification will again rely on sets of concepts $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$ consisting of all $\mathcal{EL}$-concepts D such that $\mathcal{T} \models C \sqsubseteq D$ and $\mathsf{sig}(D) \subseteq \Sigma$. However, there are two additional difficulties compared to DL-Lite: first, we do not currently know whether this approach is complete for arbitrary $\mathcal{EL}$-TBoxes. For this reason, our procedure is restricted to role-acyclic TBoxes. Second, even for role-acyclic TBoxes, explicit definitions can be of exponential size. Even worse and as shown by the following example, this problem actually manifests itself in realizations of finest Δ-decompositions, which can also be of exponential size.

Example 28 Let $\mathcal{T}$ consist of $A_i \equiv \exists r_i.A_{i+1} \sqcap \exists s_i.A_{i+1}$, for $0 \leq i < n$, and $A_n \equiv \top$. For

$$\Delta = \{r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1}\},$$

the finest Δ-decomposition of $\mathcal{T}$ is $\{A_0\}, \ldots, \{A_n\}$ because we can define a realization $\mathcal{T}_0, \ldots, \mathcal{T}_n$ by setting, inductively,

$$\mathcal{T}_n = \{A_n \equiv \top\},$$
$$\mathcal{T}_i = \{A_i \equiv \exists r_i.C_{i+1} \sqcap \exists s_i.C_{i+1}\},$$

where

$$C_n = \bot, \quad C_i = \exists r_i.C_{i+1} \sqcap \exists s_i.C_{i+1}.$$

This realization is of exponential size and that there does not exist any smaller realization of $\{A_0\}, \ldots, \{A_n\}$ using $\mathcal{EL}$-TBoxes: the smallest explicit definition of A_0 that does not use the symbols $\{A_1, \ldots, A_n\}$ corresponds to the concept representing the binary tree with edges s_i and r_i and is, therefore, of exponential size.

To resolve this problem, we consider realizations not in $\mathcal{EL}$ but in the extension $\mathcal{EL}^{\nu+}$ of $\mathcal{EL}$ by greatest fixpoints introduced and investigated in (Lutz, Piro, and Wolter 2010). In this language, explicit definitions are always of polynomial size, they can be computed in polynomial time, and, importantly, reasoning is still tractable. It will be convenient for us to use a syntactic variant, $\mathcal{EL}^{st}$, of $\mathcal{EL}^{\nu+}$ using *simulation quantifiers* instead of greatest fixpoints. To define $\mathcal{EL}^{st}$, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations, $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$ and Σ a signature. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ containing (d_1, d_2) is a Σ-*simulation* from $(\mathcal{I}_1, d_1)$ to $(\mathcal{I}_2, d_2)$ if

- for all concept names $A \in \Sigma$ and all $(e_1, e_2) \in S$, if $e_1 \in A^{\mathcal{I}_1}$, then $e_2 \in A^{\mathcal{I}_2}$;
- for all role names $r \in \Sigma$, all $(e_1, e_2) \in S$, and all $e_1' \in \Delta^{\mathcal{I}_1}$ with $(e_1, e_1') \in r^{\mathcal{I}_1}$, there exists $e_2' \in \Delta^{\mathcal{I}_2}$ such that $(e_2, e_2') \in r^{\mathcal{I}_2}$ and $(e_1', e_2') \in S$.

The relationship between $\mathcal{EL}$ and simulations has been investigated and employed extensively (Lutz and Wolter 2010; Lutz, Piro, and Wolter 2010). One important connection is that whenever there is a Σ-simulation from $(\mathcal{I}_1, d_1)$ to $(\mathcal{I}_2, d_2)$, then d_2 is an instance of any Σ-concept of which d_1 is an instance; the converse holds for all interpretations of finite outdegree. We now define $\mathcal{EL}^{st}$-concepts, CIs, and TBoxes by simultaneous induction as follows, see (Lutz, Piro, and Wolter 2010):

- every $\mathcal{EL}$-concept (CI, TBox) is an $\mathcal{EL}^{st}$-concept (CI, TBox);
- if C is an $\mathcal{EL}^{st}$-concept, $\mathcal{T}$ is an $\mathcal{EL}^{st}$-TBox, and Σ a signature, then $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is an $\mathcal{EL}^{st}$-concept;
- if C and D are $\mathcal{EL}^{st}$-concepts, then $C \sqsubseteq D$ is an $\mathcal{EL}^{st}$-CI; a finite set of $\mathcal{EL}^{st}$ CIs is an $\mathcal{EL}^{st}$-TBox.

The semantics of simulation operators is defined as follows. For any interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, let $d \in (\exists^{sim}\Sigma.(\mathcal{T}, C))^{\mathcal{I}}$ iff there exists a model $\mathcal{J}$ of $\mathcal{T}$ with a $d' \in C^{\mathcal{J}}$ such that there is a Γ-simulation from $(\mathcal{J}, d')$ to $(\mathcal{I}, d)$, where $\Gamma = (\mathsf{sig}(\mathcal{T}) \cup \mathsf{sig}(C)) \setminus \Sigma$.

Example 29 Consider the TBox $\mathcal{T}$ from Example 28. Then

$$\mathcal{T} \models C_i \equiv \exists^{sim}\{A_0, \ldots, A_n\}(\mathcal{T}, A_i),$$

for all $i \leq n$. Thus, one can realize $\{A_0\}, \ldots, \{A_n\}$ using the TBoxes $\mathcal{T}_i = \{A_i \equiv \exists^{sim}\{A_0, \ldots, A_n\}(\mathcal{T}, A_i)\}$.

Now consider the sets of concepts $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$. In contrast to the DL-Lite case, these sets can clearly be infinite. In the case of role-acyclic TBoxes, though, one can show that there always is a *finite* set of $\mathcal{EL}$-concepts equivalent to $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$. To avoid exponential size as in Example 28, we now show how to use simulation quantifiers to give a succinct representation of this finite set.

Even for arbitrary TBoxes, it is possible to prove that the concept $\exists^{sim}\Gamma.(\mathcal{T}, C)$, where $\Gamma = \mathsf{sig}(\mathcal{T}, C) \setminus \Sigma$, represents $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$ in the sense that

- $\mathcal{T} \models C \sqsubseteq \exists^{sim}\Gamma.(\mathcal{T}, C)$ and
- $\mathcal{T} \models \exists^{sim}\Gamma.(\mathcal{T}, C) \sqsubseteq D$ for all $D \in \mathsf{Cons}_{\mathcal{T},\Sigma}(C)$.

Since for role-acyclic TBoxes $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$ is equivalent to a finite set of $\mathcal{EL}$-concepts, we thus obtain the following.

Proposition 30 *Let $\mathcal{T}$ be a role-acyclic TBox and C an $\mathcal{EL}$-concept. For $\Gamma = \mathsf{sig}(\mathcal{T}, C) \setminus \Sigma$, the concept $\exists^{sim}\Gamma.(\mathcal{T}, C)$ is equivalent to the conjunction over all concepts in $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$.*

It follows that we can use the linear size concept $\exists^{sim}\Gamma.(\mathcal{T}, C)$ in place of $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$. The algorithm presented in Figure 5 is now almost a copy of the transformation algorithm for propositional DL-Lite$_{\text{horn}}$ in Figure 2. As reasoning in $\mathcal{EL}^{st}$ is still tractable (Lutz, Piro, and Wolter 2010), this algorithm runs in polynomial time. A detailed (and rather involved) proof of the following result is given in the full paper.

Theorem 31 *The algorithm $\mathsf{Rewrite}_{\mathcal{EL}}$ given in Fig. 5 transforms a given role-acyclic $\mathcal{EL}$-TBox satisfying (Dec) into an equivalent $\mathcal{EL}^{st}$-TBox in which every CI is Δ-indecomposable.*

Conclusion

We have established the theoretical foundations for a syntax-independent approach to signature decomposition in ontologies. Our investigation has been inspired by previous work in propositional logic, belief revision, and abstract logical calculi (Parikh 1999; Kourousias and Makinson 2007; Ponomaryov 2008). Of course, a semantic approach leads to reasoning services of higher complexity than purely syntactic approaches. Still, the results are quite promising: for many lightweight DLs, the main reasoning problem is still tractable and for expressive DLs it is not harder than subsumption checking. This shows that signature decomposition is computationally much simpler than semantically complete approaches to other modularization tasks such as module extraction, conservative extensions, and forgetting/uniform interpolation (Konev et al. 2009;

Input: $\mathcal{EL}$ TBox $\mathcal{T}$ satisfying (Dec) and signature $\Delta \subseteq \mathsf{sig}(\mathcal{T})$.

Apply exhaustively the following transformation rule to each $\alpha = C \sqsubseteq B \in \mathcal{T}$ such that $|\mathsf{sig}(\alpha) \setminus \Delta| \geq 2$.

1. **If** $\mathcal{T} \setminus \{\alpha\} \models \alpha$
2. **Then**
3. $\quad \mathcal{T} := \mathcal{T} \setminus \{\alpha\}$.
4. **Else**
5. $\quad$ **If** $\mathsf{sig}(C) \not\subseteq \Delta$, $\mathsf{sig}(D) \not\subseteq \Delta$, and $\mathcal{T} \models \exists^{sim}(\mathsf{sig}(\mathcal{T}) \setminus \Delta).(\mathcal{T}, C) \sqsubseteq D$
6. $\quad$ **Then**
7. $\quad \mathcal{T} := (\mathcal{T} \setminus \{\alpha\})$;
8. $\quad \mathcal{T} := \mathcal{T} \cup \{C \sqsubseteq \exists^{sim}(\mathsf{sig}(\mathcal{T}) \setminus \Delta).(\mathcal{T}, C)\} \cup \{\exists^{sim}(\mathsf{sig}(\mathcal{T}) \setminus \Delta).(\mathcal{T}, C) \sqsubseteq D\}$
9. $\quad$ **If** for $X \in \mathsf{sig}(C) \setminus \Delta$ and $\Gamma = \{X\} \cup \mathsf{sig}(\mathcal{T}) \setminus (\Delta \cup \mathsf{sig}(C))$
10. $\quad \mathcal{T} \setminus \{\alpha\} \models \exists^{sim}\Gamma.(\mathcal{T} \setminus \{\alpha\}, C) \sqsubseteq C$
11. $\quad$ **Then**
12. $\quad \mathcal{T} := (\mathcal{T} \setminus \{\alpha\}) \cup \{\exists^{sim}\Gamma.(\mathcal{T} \setminus \{\alpha\}, C) \sqsubseteq D\}$

Figure 5: Procedure $\mathsf{Rewrite}_{\mathcal{EL}}$

Lutz, Walther, and Wolter 2007; Cuenca Grau et al. 2008). Future work will include decomposition experiments with existing ontologies and the development of guidelines to determine meaningful Δ's.

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Proofs for Section "Signature Decompositions"

For the convenience of the reader, we formulate the results to be proved again.

Theorem 5 Let $\mathcal{T} \subseteq_{fin} SO$, $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, and let $\Sigma_1, \ldots, \Sigma_n$ and $\Pi_1, \ldots, \Pi_m$ be Δ-decompositions of $\mathcal{T}$ in SO. Then the Δ-decomposition with signatures $\Sigma_i \cap \Pi_j$ for all i, j with $\Sigma_i \cap \Pi_j \neq \emptyset$ is also a Δ-decomposition of $\mathcal{T}$ in SO. Thus, there exists a unique finest Δ-decomposition of $\mathcal{T}$ in SO.

Proof. Let $\mathcal{T}$, Δ, $\Sigma_1, \ldots, \Sigma_n$ and $\Pi_1, \ldots, \Pi_m$ be as in the lemma and let $\Omega_1, \ldots, \Omega_k$ be an enumeration of the signatures $\Sigma_i \cap \Pi_j$ for those i, j with $\Sigma_i \cap \Pi_j \neq \emptyset$. Clearly, $\Omega_1, \ldots, \Omega_k$ is a partition of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$. By Theorem 4, it suffices to show that

$$\exists \overline{\overline{\Omega_1}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi \wedge \cdots \wedge \exists \overline{\overline{\Omega_k}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi \models \mathcal{T} \qquad (\dagger)$$

To prove (†), let $\mathcal{I}$ be a model of the left-hand side of this entailment. It is enough to show that $\mathcal{I} \models \exists \overline{\overline{\Sigma_i}}. \bigwedge_{\varphi \in \mathcal{T}}$ for $1 \leq i \leq n$ because then, the fact that $\Sigma_1, \ldots, \Sigma_n$ is a Δ-decomposition of $\mathcal{T}$ and Theorem 4 yield $\mathcal{I} \models \mathcal{T}$.

Fix a Σ_i. By choice of $\mathcal{I}$, there is a model $\mathcal{I}_{i,j}$ of $\mathcal{T}$ that agrees with $\mathcal{I}$ on the interpretation of $\Sigma_i \cap \Pi_j$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$ with $\Sigma_i \cap \Pi_j \neq \emptyset$. Let $j_1, \ldots, j_p \in \{1, \ldots, m\}$ be the set of those indices j_ℓ such that $\Sigma_i \cap \Pi_{j_\ell} \neq \emptyset$. Now let $\mathcal{J}$ be the interpretation that agrees

- with $\mathcal{I}_{i,j_\ell}$ on the interpretation of all symbols from Π_{j_ℓ}, for $1 \leq \ell \leq p$;
- with some $\mathcal{I}_{x,y}$ on the interpretation of all symbols from Π_j, for all Π_j with $\Sigma_i \cap \Pi_j = \emptyset$.

To show $\mathcal{I} \models \exists \overline{\overline{\Sigma_i}}. \bigwedge_{\varphi \in \mathcal{T}}$, it clearly suffices to prove that

1. $\mathcal{I}$ and $\mathcal{J}$ agree on the interpretation of all symbols in Σ_i;
2. $\mathcal{J} \models \mathcal{T}$.

For Point 1, first note that $\Sigma_i = \bigcup_{1 \leq \ell \leq p} (\Sigma_i \cap \Pi_{j_\ell})$. Thus, it suffices to show that $\mathcal{I}$ and $\mathcal{J}$ agree on the interpretation of all symbols in $\Sigma_i \cap \Pi_{j_\ell}$, for $1 \leq \ell \leq p$. But this is clearly since $\mathcal{I}$ and $\mathcal{I}_{i,j_\ell}$ agree on the interpretation of all symbols in $\Sigma_i \cap \Pi_{j_\ell}$, and so do $\mathcal{I}_{i,j_\ell}$ and $\mathcal{J}$. For Point 2, recall that $\Pi_1, \ldots, \Pi_m$ is a Δ-decomposition of $\mathcal{T}$. By definition of $\mathcal{J}$, the models $\mathcal{I}_{i,j}$ of $\mathcal{T}$ show that $\mathcal{J} \models \exists \overline{\overline{\Pi_j}}. \bigwedge_{\varphi \in \mathcal{T}} \varphi$. By Theorem 4, we thus get $\mathcal{J} \models \mathcal{T}$. ❑

Theorem 8 SO has unique decomposition realizations.

Proof. Let $\mathcal{T} \subseteq_{\mathsf{fin}}$ SO, $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, $\Sigma_1, \ldots, \Sigma_n$ a Δ-decomposition of $\mathcal{T}$ in SO, and $\mathcal{T}_1, \ldots, \mathcal{T}_n$ and $\mathcal{T}_1', \ldots, \mathcal{T}_n'$ realizations of $\Sigma_1, \ldots, \Sigma_n$ in SO such that

- $\mathcal{T}_i, \mathcal{T}_j$ are Δ-inseparable w.r.t. SO for $i, j \leq n$ and
- $\mathcal{T}_i', \mathcal{T}_j'$ are Δ-inseparable w.r.t. SO for $i, j \leq n$.

Fix an $i \leq n$. We have to show that $\mathcal{T}_i \models \mathcal{T}_i'$ and $\mathcal{T}_i' \models \mathcal{T}_i$, and concentrate on the former since the latter is symmetric. Let $\mathcal{I}$ be a model of $\mathcal{T}_i$. Then all of $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are satisfiable: if $\mathcal{T}_j$ was unsatisfiable, then $\mathcal{T}_j \models F$ (where F abbreviates $\exists x.(x \neq x)$), which implies $\mathcal{T}_i \models F$ since $\mathcal{T}_i$ and $\mathcal{T}_j$ are Δ-inseparable w.r.t. SO and $\mathsf{sig}(F) = \emptyset \subseteq \Delta$, which in turn contradicts the existence of $\mathcal{I}$. It is shown in (Konev et al. 2009) that satisfiability of $\mathcal{T}_1, \ldots, \mathcal{T}_n$ together with their Δ-inseparability implies that there are models $\mathcal{I}_1, \ldots, \mathcal{I}_n$ of $\mathcal{T}_1, \ldots, \mathcal{T}_n$, respectively, that satisfy

- $\mathcal{I}_1|_\Delta = \cdots = \mathcal{I}_n|_\Delta$, where $\mathcal{I}_i|_\Delta$ denotes the restriction of $\mathcal{I}_i$ to only the symbols in Δ;
- $\mathcal{I}_i = \mathcal{I}$.

Let $\mathcal{J}$ be the model obtained by interpreting all symbols from $\Sigma_i \cup \Delta$ as in $\mathcal{I}_i$. Since $\mathsf{sig}(\mathcal{T}_i) \subseteq \Sigma_i \cup \Delta$, $\mathcal{J}$ is clearly a model of $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$. Thus, it is also a model of $\mathcal{T}$, whence of $\mathcal{T}_1', \ldots, \mathcal{T}_n'$. Since $\mathsf{sig}(\mathcal{T}_i') \subseteq \Sigma_i \cup \Delta$, and $\mathcal{J}|_{\Sigma_i \cup \Delta} = \mathcal{I}_i|_{\Sigma_i \cup \Delta} = \mathcal{I}|_{\Sigma_i \cup \Delta}$, $\mathcal{I}$ must also be a model of $\mathcal{T}_i'$ as required. ❑

Proofs for "Signature decompositions and parallel interpolation in DLs"

Theorem 12 Let $\mathcal{L}$ be a fragment of SO with the parallel interpolation property. Then

- $\mathcal{L}$-decompositions coincide with SO-decompositions.
- $\mathcal{L}$-decompositions have unique realizations.

In particular, there always exists a unique finest signature Δ-decomposition in $\mathcal{L}$.

Proof. Point 1. We prove this result for Δ-decompositions into two signatures. The generalization to Δ-decompositions into $n > 2$ signatures is straightforward and left to the reader. Assume $\mathcal{T}$ in $\mathcal{L}$ and Δ are given. Assume that Σ_1, Σ_2 is a Δ-decomposition in SO of $\mathcal{T}$. It follows from Theorem 4 that $\{\exists \Sigma_2. \bigwedge_{\varphi \in \mathcal{T}} \varphi, \exists \Sigma_1. \bigwedge_{\varphi \in \mathcal{T}} \varphi\} \models \mathcal{T}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the subsets of $\mathcal{L}$ obtained from $\mathcal{T}$ by replacing all predicates in Σ_2 by fresh predicates in Σ_2^* and all predicates in Σ_1 by fresh predicates in Σ_1^*. Then $\mathcal{S}_1 \cup \mathcal{S}_2 \models \mathcal{T}$. Moreover, $\mathcal{S}_1$ and $\mathcal{S}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$ and $\mathsf{sig}(\mathcal{S}_1) \cap \mathsf{sig}(\mathcal{S}_2) \subseteq \Delta$. Thus, by the PIP of $\mathcal{L}$, there exist, for every $\alpha \in \mathcal{T}$, parallel interpolants $(\mathcal{T}_\alpha^1, \mathcal{T}_\alpha^2)$ in $\mathcal{L}$ for $(\mathcal{S}_1, \mathcal{S}_2)$ and α. Now let $\mathcal{T}_i' = \bigcup_{\alpha \in \mathcal{T}} \mathcal{T}_\alpha^i$ for $i = 1, 2$. It is not difficult to show that $\mathcal{T}_1', \mathcal{T}_2'$ is a realization of Σ_1, Σ_2 in $\mathcal{L}$.

Point 2. Again we prove this for Δ-decompositions into two signatures. Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_1', \mathcal{T}_2'$ realize a Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$ and that $\mathcal{T}_1, \mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$ and $\mathcal{T}_1', \mathcal{T}_2'$ are Δ-inseparable w.r.t. $\mathcal{L}$. Assume w.l.o.g. that $\mathcal{T}_1 \models \alpha$ but $\mathcal{T}_1' \not\models \alpha$. We may assume that $\mathsf{sig}(\alpha) \subseteq \Sigma_1 \cup \Delta$. But then $\mathcal{T}_1' \cup \mathcal{T}_2' \models \alpha$ because $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$. It follows from the PIP of $\mathcal{L}$ that $\mathcal{T}_1' \models \alpha$ (see below for a proof of the RJCP from PIP), and we have derived a contradiction. ❑

To analyze the PIP and prove the results of this section, we connect it to two properties previously studied. The first is the Robinson Joint Consistency Property, the second a Boolean variant of the Craig Interpolation Property.

Definition 32 (Robinson Joint Consistency Property) *Let $\mathcal{L}$ be a fragment of FO. $\mathcal{L}$ has the Robinson Joint Consistency Property (RJCP) if the following holds for any two (possibly infinite) $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{L}$: if $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$, then*

$$\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha \quad \Leftrightarrow \quad \mathcal{T}_1 \models \alpha$$

for all sentences α in $\mathcal{L}$ with $\mathsf{sig}(\alpha) \subseteq \mathsf{sig}(\mathcal{T}_1)$.

Definition 33 (Boolean Craig Interpolation) *Let $\mathcal{L}$ be a fragment of FO. $\mathcal{L}$ has Boolean Craig Interpolation if for all $\mathcal{T} \subseteq \mathcal{L}$ and all Boolean combinations φ of $\mathcal{L}$-sentences the following holds: if*

$$\mathcal{T} \models \varphi,$$

then there exists a Boolean combination ψ of $\mathcal{L}$-sentences with $\mathsf{sig}(\psi) \subseteq \mathsf{sig}(\mathcal{T}) \cap \mathsf{sig}(\varphi)$ such that $\mathcal{T} \models \psi$ and $\psi \models \varphi$.

We say that a fragment $\mathcal{L}$ of FO has the *disjoint union property* if the following holds for all $\mathcal{T} \subseteq \mathcal{L}$: for all families $\mathcal{I}_i$, $i \in I$, of interpretations the following conditions are equivalent:

- all $\mathcal{I}_i$, $i \in I$, are models of $\mathcal{T}$;
- the disjoint union of all $\mathcal{I}_i$, $i \in I$, is a model of $\mathcal{T}$.

Note that $\mathcal{EL}$, $\mathcal{ELH}$, $\mathcal{ALC}$, $\mathcal{ALCQI}$ and all standard dialects of DL-Lite have the disjoint union property. Examples of DLs without the disjoint union property are DLs with nominals or the universal role. We will make frequent use of the following property of languages $\mathcal{L}$ with the disjoint union property: if

$$\mathcal{T} \models \bigwedge_{i \in I} \alpha_i \rightarrow \bigvee_{j \in J} \beta_j$$

where the α_i and β_j are $\mathcal{L}$-sentences and $\mathcal{T}$ is a set of $\mathcal{L}$-sentences, then there exists $j \in J$ such that

$$\mathcal{T} \models \bigwedge_{i \in I} \alpha_i \rightarrow \beta_j.$$

Theorem 34 *Let $\mathcal{L} \subseteq FO$ have the disjoint union property. Then the following conditions are equivalent:*

1. *$\mathcal{L}$ has the Parallel Interpolation Property;*
2. *$\mathcal{L}$ has RJCP;*
3. *$\mathcal{L}$ has the Boolean Craig Interpolation Property.*

Proof. (1) implies (2). Assume $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$, $\mathcal{T}_1, \mathcal{T}_2$ are Δ-inseparable, $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, and $\mathsf{sig}(\alpha) \subseteq \mathcal{T}_1$.

Take Δ-parallel interpolants $\mathcal{T}_1', \mathcal{T}_2'$ of $\mathcal{T}_1, \mathcal{T}_2$ and α in $\mathcal{L}$. Then $\mathsf{sig}(\mathcal{T}_2') \subseteq \Delta$. Hence, by Δ-inseparability w.r.t. $\mathcal{L}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ and because $\mathcal{T}_2 \models \mathcal{T}_2'$, we obtain $\mathcal{T}_1 \models \mathcal{T}_2'$. Hence, $\mathcal{T}_1 \models \mathcal{T}_1' \cup \mathcal{T}_2'$ and so from $\mathcal{T}_1' \cup \mathcal{T}_2' \models \alpha$ we obtain $\mathcal{T}_1 \models \alpha$, as required.

We now prove the equivalence of (2) and (3).

(2) implies (3). Assume that $\mathcal{L}$ has RJCP and let $\mathcal{T} \models \varphi$. We construct an interpolant. By the disjoint union property we may assume w.l.o.g. that

$$\varphi = \bigwedge_{i \in I} \alpha_i \rightarrow \beta$$

where the α_i and β are $\mathcal{L}$-sentences and $\mathsf{sig}(\beta) \subseteq \mathsf{sig}(\bigwedge_{i \in I} \alpha_i)$.

Let $\Delta = \mathsf{sig}(\mathcal{T}) \cap \mathsf{sig}(\varphi)$. Assume there does not exist a Boolean combination ψ of $\mathcal{L}$-sentences with $\mathsf{sig}(\psi) \subseteq \Delta$ such that $T \models \psi$ and $\psi \models \varphi$. In what follows we set $\bigwedge S = \bigwedge_{\psi \in S} \psi$ whenever S is a finite set of $\mathcal{L}$-sentences. Let T_Δ denote the set of all sentences $\bigwedge T' \rightarrow \alpha$ such that

- $\mathcal{T} \models \bigwedge T' \rightarrow \alpha$;
- $\mathsf{sig}(\bigwedge T' \rightarrow \alpha) \subseteq \Delta$;
- T' is a finite set of $\mathcal{L}$-sentences;
- α is an $\mathcal{L}$-sentence.

By compactness and the disjoint union property we have $T_\Delta \not\models \varphi$. Take a model $\mathcal{I}$ of T_Δ and $\neg\varphi$. Let

$$Th^+(\mathcal{I}) = \{\gamma \mid \mathcal{I} \models \gamma, \gamma \text{ a } \mathcal{L}\text{-sentence with } \mathsf{sig}(\gamma) \subseteq \Delta\},$$

$$Th^-(\mathcal{I}) = \{\neg\gamma \mid \mathcal{I} \not\models \gamma, \gamma \text{ a } \mathcal{L}\text{-sentence with } \mathsf{sig}(\gamma) \subseteq \Delta\},$$

and set

$$Th(\mathcal{I}) = Th^+(\mathcal{I}) \cup Th^-(\mathcal{I}).$$

Then $Th(\mathcal{I}) \cup \mathcal{T}$ is satisfiable, by the disjoint union property. For suppose not. Then there are $\gamma_1 \dots, \gamma_n \in Th^+(\mathcal{I})$ and $\neg\gamma_{n+1}, \dots, \neg\gamma_m \in Th^-(\mathcal{I})$ such that

$$\mathcal{T} \models (\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow (\gamma_{n+1} \vee \cdots \vee \gamma_m).$$

By the disjoint union property, there exists γ_j, $n < j \leq m$, such that

$$\mathcal{T} \models (\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \gamma_j.$$

But then $(\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \gamma_j \in T_\Delta$ which is a contradiction because $\mathcal{I}$ is a model of T_Δ and $\mathcal{I} \models \gamma_1 \wedge \ldots \wedge \gamma_n$ but $\mathcal{I} \not\models \gamma_j$. It follows that if $\mathcal{T} \cup Th^+(\mathcal{I}) \models \gamma$, then $\gamma \in Th^+(\mathcal{I})$ for all $\mathcal{L}$-sentences γ such that $\mathsf{sig}(\gamma) \subseteq \Delta$. Hence

$$\mathcal{T} \cup Th^+(\mathcal{I}), \quad Th^+(\mathcal{I}) \cup \{\alpha_i \mid i \in I\}$$

are Δ-inseparable w.r.t. $\mathcal{L}$. We clearly have

$$\mathcal{T} \cup Th^+(\mathcal{I}) \quad \cup \quad Th^+(\mathcal{I}) \cup \{\alpha_i \mid i \in I\} \models \beta.$$

Hence, by RJCP, $Th^+(\mathcal{I}) \cup \{\alpha_i \mid i \in I\} \models \beta$. We have derived a contradiction since $\neg\varphi = \bigwedge_{i \in I} \alpha_i \wedge \neg\beta$ is satisfied in $\mathcal{I}$.

(3) implies (2).

Assume $\mathcal{T}_1 \cup \mathcal{T}_2 \models \varphi$, $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{L}$, and $\mathsf{sig}(\varphi) \subseteq \mathsf{sig}(\mathcal{T}_1)$. We show that $\mathcal{T}_1 \models \varphi$. By compactness, we may assume that $\mathcal{T}_1, \mathcal{T}_2$ are finite. Then we have

$$\mathcal{T}_2 \models \bigwedge \mathcal{T}_1 \rightarrow \varphi.$$

By Boolean Craig Interpolation, there is a Boolean combination ψ of $\mathcal{L}$-sentences such that

$$\mathcal{T}_2 \models \psi, \quad \psi \models \bigwedge \mathcal{T}_1 \rightarrow \varphi$$

and $\mathsf{sig}(\psi) \subseteq \Delta$. We have

$$\mathcal{T}_1 \models \psi \rightarrow \varphi.$$

Now, assume that $\mathcal{T}_1 \not\models \varphi$. Take a model $\mathcal{I}$ of $\mathcal{T}_1$ refuting φ. We show that this leads to a contradiction. By the disjoint union property, we may assume that $\mathcal{I}$ refutes every

$\mathcal{L}$-sentence β with $\mathsf{sig}(\beta) \subseteq \Delta$ such that $\mathcal{T}_1 \not\models \beta$. (If not, take for every such β with a model $\mathcal{I}_\beta$ of $\mathcal{T}_1$ that refutes β and take the disjoint union of all $\mathcal{I}_\beta$ and $\mathcal{I}$.)

We now show that $\mathcal{I} \models \psi$, and thus derive a contradiction to $\mathcal{T}_1 \models \psi \rightarrow \varphi$. Assume $\mathcal{I} \not\models \psi$. By the disjoint union property, we may assume w.l.o.g. that ψ is a conjunction of formulas of the form

$$\bigwedge \alpha_i \rightarrow \beta$$

where the α_i and β are $\mathcal{L}$-sentences. Thus, we obtain a conjunct

$$\chi = \bigwedge \alpha_i \rightarrow \beta$$

of ψ with the properties

- $\mathcal{T}_2 \models \chi$;
- $\mathsf{sig}(\chi) \subseteq \Delta$;
- $\mathcal{I} \not\models \chi$.

It follows that $\mathcal{I} \models \bigwedge \alpha_i$. By construction of $\mathcal{I}$, $\mathcal{I} \models \bigwedge \alpha_i$. By Δ-inseparability of $\mathcal{T}_1$ and $\mathcal{T}_2$ w.r.t. $\mathcal{L}$, $\mathcal{T}_2 \models \bigwedge \alpha_i$. Hence, since $\mathcal{T}_2 \models \chi$, $\mathcal{T}_2 \models \beta$. By Δ-inseparability of $\mathcal{T}_1$ and $\mathcal{T}_2$ w.r.t. $\mathcal{L}$, $\mathcal{T}_1 \models \beta$. Hence $\mathcal{I} \models \beta$ and so $\mathcal{I} \models \chi$ and we have a contradiction.

((2) and (3)) implies (1).

Assume $\mathcal{T}_1$, $\mathcal{T}_2$, α_0, and Δ are given and satisfy the preconditions 1.-3. for the parallel interpolation property. By compactness, we may assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are finite. We have

$$\mathcal{T}_1 \models \bigwedge \mathcal{T}_2 \rightarrow \alpha_0.$$

Thus, there exists a Boolean Craig Interpolant ψ_1 for $\mathcal{T}_1$ and $\bigwedge \mathcal{T}_2 \rightarrow \alpha_0$). We have

$$\mathcal{T}_1 \models \psi_1, \quad \psi_1 \models \bigwedge \mathcal{T}_2 \rightarrow \alpha_0.$$

By the disjoint union property, we may assume that ψ_1 is a conjunction of formulas of the form

$$\bigwedge \alpha_i \rightarrow \beta,$$

where α_i and β are $\mathcal{L}$-sentences.

The consequence

$$\psi_1 \models \bigwedge \mathcal{T}_2 \rightarrow \alpha_0.$$

is equivalent to

$$\mathcal{T}_2 \models \psi_1 \rightarrow \alpha_0.$$

Again, by Boolean Craig Interpolation, we find an interpolant ψ_2 for $\mathcal{T}_2$ and $\psi_1 \rightarrow \alpha_0$. We may assume that ψ_2 is of the same form as ψ_1 and have

$$\mathcal{T}_2 \models \psi_2, \quad \psi_2 \models (\psi_1 \rightarrow \alpha_0).$$

Summarizing, we have

- $\mathcal{T}_1 \models \psi_1, \mathcal{T}_2 \models \psi_2$;
- $\{\psi_1, \psi_2\} \models \alpha_0$;
- $\mathsf{sig}(\psi_i) \setminus \Delta \subseteq \mathsf{sig}(\mathcal{T}_i) \cap \mathsf{sig}(\alpha_0)$, for $i = 1, 2$.

Thus ψ_1, ψ_2 satisfy the conditions for parallel interpolants except that they are not $\mathcal{L}$-sentences but Boolean combinations of $\mathcal{L}$-sentences.

Set $\Gamma_i = \Delta \cup (\mathsf{sig}(\alpha_0) \cap \mathsf{sig}(\mathcal{T}_i))$ and let

$$\mathcal{T}_i' = \{\alpha \mid \mathcal{T}_i \models \alpha, \alpha \text{ a } \mathcal{L}\text{-sentence}, \mathsf{sig}(\alpha) \subseteq \Gamma_i\},$$

for $i = 1, 2$. We show that $\mathcal{T}_1', \mathcal{T}_2'$ is a parallel interpolant. The only non-trvial condition is

$$\mathcal{T}_1' \cup \mathcal{T}_2' \models \alpha_0.$$

Assume that this is not the case. Take a model $\mathcal{I}$ of $\mathcal{T}_1' \cup \mathcal{T}_2'$ such that $\mathcal{I}$ refutes α_0. We may assume that $\mathcal{I}$ refutes all $\mathcal{L}$-sentences β such that

- $\mathcal{T}_1' \not\models \beta$ and $\mathsf{sig}(\beta) \subseteq \Gamma_1$;
- $\mathcal{T}_2' \not\models \beta$ and $\mathsf{sig}(\beta) \subseteq \Gamma_2$.

This follows from the disjoint union property and RJCP (because $\mathcal{T}_1' \cup \mathcal{T}_2' \not\models \beta$ for such β). We show that $\mathcal{I}$ is a model of ψ_1, ψ_2, thus obtaining a contradiction. Let

$$\varphi = \bigwedge \alpha_i \rightarrow \beta$$

be a conjunct of ψ_1 and assume $\mathcal{I} \not\models \varphi$. It follows that $\mathcal{I} \models \bigwedge \alpha_i$. By construction of $\mathcal{I}$, $\mathcal{T}_1' \models \bigwedge \alpha_i$. Hence $\mathcal{T}_1 \models \bigwedge \alpha_i$ and so $\mathcal{T}_1 \models \beta$. But then $\mathcal{T}_1' \models \beta$ from which we obtain $\mathcal{I} \models \beta$ and, therefore, $\mathcal{I} \models \varphi$. We have derived a contradiction. The same argument can be used to prove that $\mathcal{I} \models \psi_2$. ❑

Using Theorem 34 we can now prove Theorem 13 for expressive DLs by applying the result of (Konev et al. 2009) that $\mathcal{ALC}$, $\mathcal{ALCQ}$, $\mathcal{ALCI}$, $\mathcal{ALCQI}$ have the Boolean Craig Interpolation property.

Theorem 35 *$\mathcal{ALC}$, $\mathcal{ALCQ}$, $\mathcal{ALCI}$, $\mathcal{ALCQI}$ have Boolean Craig Interpolation and, therefore, the PIP.*

The $\mathcal{EL}$ and $\mathcal{ELH}$-part of Theorem 13 follows from Theorem 34 and the following result:

Theorem 36 *$\mathcal{EL}$ and $\mathcal{ELH}$ have RJCP.*

Proof. For $\mathcal{EL}$, this has been proved in (Lutz and Wolter 2010). Here we present the proof for $\mathcal{ELH}$. Note that this is a non-trivial extension because, as we have seen for $\mathcal{ALCH}$, the addition of role inclusions can easily lead to logics without RJCP.

Assume TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathcal{ELH}$ are given, they are Σ-inseparable w.r.t. $\mathcal{ELH}$ for $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Sigma$, and $\mathcal{T}_1 \not\models \alpha$, where $\mathsf{sig}(\alpha) \subseteq \mathsf{sig}(\mathcal{T}_1)$. We show that $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models \alpha$.

First consider $\alpha = r_0 \sqsubseteq s_0$. Let $\mathcal{R}_i = \{r \sqsubseteq s \mid r \sqsubseteq s \in \mathcal{T}_i\}$.

Claim. $\mathcal{T}_1 \cup \mathcal{T}_2 \models r \sqsubseteq s$ iff $r, s \in \mathsf{sig}(\mathcal{R}_i)$ and $\mathcal{R}_i \models r \sqsubseteq s$ for some $i = 1, 2$ or there exists $s' \in \Delta$ such that $\mathcal{R}_i \models r \sqsubseteq s'$ and $\mathcal{R}_{i'} \models s' \sqsubseteq s$, where $\{i, i'\} = \{1, 2\}$.

The proof is straightforward and left to the reader. It follows immediately that $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models r_0 \sqsubseteq s_0$, as required.

Now assume that $\alpha = C_0 \sqsubseteq D_0$. We show $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models C_0 \sqsubseteq D_0$.

In what follows we write

$$\mathcal{T} \cup \Xi \models C$$

for a TBox $\mathcal{T}$, a set Ξ of $\mathcal{EL}$-concepts, and a $\mathcal{EL}$-concept C if in every model $\mathcal{I}$ of $\mathcal{T}$ and $d \in \Delta^{\mathcal{I}}$ such that $d \in D^{\mathcal{I}}$ for all $D \in \Xi$, we have $d \in C^{\mathcal{I}}$. Take a model $\mathcal{I}_0$ of $\mathcal{T}_1$ with $d_0 \in \Delta^{\mathcal{I}_0}$ such that for all $\mathcal{EL}$-concepts C the following holds:

$$d_0 \in C^{\mathcal{I}_0} \text{ iff } \mathcal{T}_1 \cup \{C_0\} \models C$$

Such an interpretation exists by (Lutz and Wolter 2010). Then $d_0 \in C_0^{\mathcal{I}_0} \setminus D_0^{\mathcal{I}_0}$. Set $\Delta_0 = \Delta_{d_0} = \Delta^{\mathcal{I}_0}$. In the following, we construct an interpretation $\mathcal{I}^*$ of $\mathcal{T}_1 \cup \mathcal{T}_2$ expanding $\mathcal{I}_0$ such that $d_0 \in C_0^{\mathcal{I}^*}$ and $d_0 \notin D_0^{\mathcal{I}^*}$. We define inductively an infinite sequence $\mathcal{I}_1, \mathcal{I}_2, \ldots$ of interpretations. The interpretation $\mathcal{I}^* = (\Delta^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is then defined as the union of $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ as follows:

$$\begin{array}{rcl} \Delta^{\mathcal{I}^*} & := & \bigcup_{i \geq 0} \Delta^{\mathcal{I}_i}; \\ A^{\mathcal{I}^*} & := & \bigcup_{i \geq 0} A^{\mathcal{I}_i}, \text{ for all } A \in \mathsf{N_C}; \\ r^{\mathcal{I}^*} & := & \bigcup_{i \geq 0} r^{\mathcal{I}_i}, \text{ for all } r \in \mathsf{N_R}. \end{array}$$

Given an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, let $d^{\Sigma,\mathcal{I}}$ denote the set of $\mathcal{EL}$-concepts E such that $\mathsf{sig}(E) \subseteq \Sigma$ and $d \in E^{\mathcal{I}}$. For any TBox $\mathcal{T}$ denote by $\mathcal{I}_{t_{\mathcal{I}}(d),\mathcal{T}}$ a model of $\mathcal{T}$ with d in its domain such that

$(*)$ $d \in E^{\mathcal{I}_{t_{\mathcal{I}}(d),\mathcal{T}}}$ iff $\mathcal{T} \cup d^{\Sigma,\mathcal{I}} \models E$, for all $\mathcal{EL}$-concepts E.

Such an interpretation always exists, see (Lutz and Wolter 2010). Moreover, we may assume that d is not within the range of any $r^{\mathcal{I}_{t_{\mathcal{I}}(d),\mathcal{T}}}$ (if it is, one can use standard unravelling to obtain a model with the required properties). Let $n \geq 0$ and assume the interpretation $\mathcal{I}_n$ with domain Δ_n has been defined. If n is even, then take for every $d \in \Delta_n \setminus \Delta_{n-1}$ (we set $\Delta_{-1} = \emptyset$) the interpretation $\mathcal{I}_d = \mathcal{I}_{t_{\mathcal{I}_n}(d),\mathcal{T}_2}$ with domain Δ_d such that $\Delta_n \cap \Delta_d = \{d\}$ and the Δ_d, $d \in \Delta_n \setminus \Delta_{n-1}$, are mutually disjoint. If n is odd, then take for every $d \in \Delta_n \setminus \Delta_{n-1}$ the interpretation $\mathcal{I}_d = \mathcal{I}_{t_{\mathcal{I}_n}(d),\mathcal{T}_1}$ with domain Δ_d such that $\Delta_n \cap \Delta_d = \{d\}$ and the Δ_d, $d \in \Delta_n \setminus \Delta_{n-1}$, are mutually disjoint. Now set

$$\begin{array}{rcl} \Delta_{n+1} & = & \Delta_n \cup \bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} \Delta_d, \\ A^{\mathcal{I}_{n+1}} & = & A^{\mathcal{I}_n} \cup \bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} A^{\mathcal{I}_d}. \end{array}$$

and

$$r^{\mathcal{I}_{n+1}} = r^{\mathcal{I}_n} \cup \bigcup_{\mathcal{T}_1 \cup \mathcal{T}_2 \models s \sqsubseteq r} s^{\mathcal{I}_n} \cup \bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} r^{\mathcal{I}_d}.$$

For all $d \in \Delta^{\mathcal{I}^*}$ there exists a (uniquely) determined minimal natural number $n(d)$ with $d \in \Delta_{n(d)} \setminus \Delta_{n(d)-1}$. If $n(d) \neq 0$, then there exists a uniquely determined $d^* \in \Delta_{n(d)-1}$ with $d \in \Delta_{d^*}$. We set $d^* = d_0$ for $n(d) = 0$ and prove the following by induction on the construction of D. For all $d \in \Delta^{\mathcal{I}^*}$ and $\mathcal{EL}$-concepts D:

- if $n(d)$ is even then
  1. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1) \subseteq \Sigma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_d}$;
  2. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Sigma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_{d^*}}$;
- if $n(d)$ is odd then
  1. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Sigma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_d}$;
  2. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1) \subseteq \Sigma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_{d^*}}$.

The implications from right to left are trivial, so we consider the implications from left to right only. We concentrate on the case $n(d)$ even (the case $n(d)$ odd is proved in the same way) and prove the induction step for $D = \exists r.C$. First consider Point 1. So let $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1) \subseteq \Sigma$ and assume $d \in D^{\mathcal{I}^*}$ with $n(d)$ even. There exists $c \in \Delta^{\mathcal{I}^*}$ such that $c \in C^{\mathcal{I}^*}$ and $(d,c) \in r^{\mathcal{I}^*}$. Assume first that $c \in \Delta_{n(d)}$. Then, by construction,

- $r \in \mathsf{sig}(\mathcal{T}_1)$ (and so $r \in \Sigma$); or
- $r \notin \mathsf{sig}(\mathcal{T}_1)$ and there exists $s \in \Sigma$ such that $\mathcal{T}_1 \cup \mathcal{T}_2 \models s \sqsubseteq r$ and $(d,c) \in s^{\mathcal{I}^*}$.

We obtain $n(c) = n(d)$ and, by IH, $c \in C^{\mathcal{I}_c}$. We obtain $\mathcal{T}_2 \cup c^{\Sigma,\mathcal{I}_{n(d)}} \models C$. By compactness and closure under conjunction of $c^{\Sigma,\mathcal{I}_{n(d)}}$, there exists a concept C_0 in $c^{\Sigma,\mathcal{I}_{n(d)}}$ with $\mathcal{T}_2 \models C_0 \sqsubseteq C$.

If $r \in \mathsf{sig}(\mathcal{T}_1)$, then we get $\mathcal{T}_2 \models \exists r.C_0 \sqsubseteq \exists r.C$. We have $\exists r.C_0 \in d^{\Sigma,\mathcal{I}_{n(d)}}$ and so $\mathcal{T}_2 \cup d^{\Sigma,\mathcal{I}_{n(d)}} \models \exists r.C$. But then $d \in D^{\mathcal{I}_d}$.

If $r \notin \mathsf{sig}(\mathcal{T}_1)$, we have $\mathcal{T}_2 \models \exists s.C_0 \sqsubseteq \exists s.C$. We have $\exists s.C_0 \in d^{\Sigma,\mathcal{I}_{n(d)}}$ and so, because $\mathcal{T}_2 \models s \sqsubseteq r$, $\mathcal{T}_2 \cup d^{\Sigma,\mathcal{I}_{n(d)}} \models \exists r.C$. But then $d \in D^{\mathcal{I}_d}$.

Now assume $c \notin \Delta_{n(d)}$. Then $c \in \Delta_d$, $c^* = d$, and $n(c) = n(d) + 1$. By induction hypothesis (for $n(c)$ odd), $c \in C^{\mathcal{I}^*}$ iff $c \in C^{\mathcal{I}_{c^*}} = C^{\mathcal{I}_d}$. Hence $d \in (\exists r.C)^{\mathcal{I}_d}$.

Consider now Point 2. Let $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Sigma$ and $d \in D^{\mathcal{I}^*}$. There exists $c \in \Delta^{\mathcal{I}^*}$ such that $c \in C^{\mathcal{I}^*}$ and $(d,c) \in r^{\mathcal{I}^*}$. Assume first that $c \in \Delta_{d^*}$. Then $c^* = d^*$ and, by induction hypothesis, $c \in C^{\mathcal{I}_{d^*}}$. As we also have $(d,c) \in r^{\mathcal{I}_{d^*}}$, we obtain $d \in D^{\mathcal{I}_{d^*}}$.

Now assume $c \notin \Delta_{d^*}$. Then $c \in \Delta_d$. Then

- $r \in \Sigma$ or
- $r \notin \Sigma$ and there exists $s \in \Sigma$ with $\mathcal{T}_1 \cup \mathcal{T}_2 \models s \sqsubseteq r$ and $(d,c) \in s^{\mathcal{I}^*}$

By induction hypothesis $c \in C^{\mathcal{I}_c}$. Hence $\mathcal{T}_1 \cup c^{\Sigma,\mathcal{I}_{n(d)+1}} \models C$. By compactness and closure under conjunction of $c^{\Sigma,\mathcal{I}_{n(d)+1}}$, there exists a concept C_0 in $c^{\Sigma,\mathcal{I}_{n(d)+1}}$ with $\mathcal{T}_1 \models C_0 \sqsubseteq C$.

Consider the first case: Then $\mathcal{T}_1 \models \exists r.C_0 \sqsubseteq \exists r.C$. We have $d \in (\exists r.C_0)^{\mathcal{I}_d}$. Since $\mathsf{sig}(\exists r.C_0) \subseteq \Sigma$ it follows from Σ-inseparability w.r.t. $\mathcal{ELH}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ and compactness that $\exists r.C_0 \in d^{\Sigma,\mathcal{I}_{n(d)}}$. So $d \in (\exists r.C_0)^{\mathcal{I}_{d^*}}$. $\mathcal{I}_{d^*}$ is a model of $\mathcal{T}_1$. Hence $d \in (\exists r.C)^{\mathcal{I}_{d^*}}$.

Consider the second case: Then $\mathcal{T}_1 \models \exists s.C_0 \sqsubseteq \exists s.C$. We have $d \in (\exists s.C_0)^{\mathcal{I}_d}$. Since $\mathsf{sig}(\exists s.C_0) \subseteq \Sigma$ it follows from Σ-inseparability w.r.t. $\mathcal{ELH}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ and compactness that $\exists s.C_0 \in d^{\Sigma,\mathcal{I}_{n(d)}}$. So $d \in (\exists s.C_0)^{\mathcal{I}_{d^*}}$. $\mathcal{I}_{d^*}$ is a model of $\mathcal{T}_1$. Hence $d \in (\exists s.C)^{\mathcal{I}_{d^*}}$. Finally, $d \in (\exists r.C)^{\mathcal{I}_{d^*}}$ follows.

It follows immediately that $\mathcal{I}^*$ is a model of $\mathcal{T}_1 \cup \mathcal{T}_2$: first notice that, by definition, $\mathcal{I}^*$ is a model of all $r \sqsubseteq s \in \mathcal{T}_1 \cup$

$\mathcal{T}_2$. Moreover for each interpretation $\mathcal{I}_d$ constructed during the construction of $\mathcal{I}^*$ as a model of $\mathcal{T}_i$ the interpretation of symbols in $\mathsf{sig}(\mathcal{T}_i)$ does not change. Now let $C \sqsubseteq D \in \mathcal{T}_i$. If $C^{\mathcal{I}^*} \setminus D^{\mathcal{I}^*} \neq \emptyset$, then there exists a an interpretation $\mathcal{I}_d$ of $\mathcal{T}_i$ with $C^{\mathcal{I}_d} \setminus D^{\mathcal{I}_d} \neq \emptyset$ which is a contradiction.

It remains to show that $d_0 \in C_0^{\mathcal{I}^*} \setminus D_0^{\mathcal{I}^*}$. $d_0 \in C^{\mathcal{I}^*}$ by the claim above and since $d_0 \in C^{\mathcal{I}_0}$. $d_0 \notin D^{\mathcal{I}^*}$ follows from $d_0 \notin D^{\mathcal{I}_0}$ and the claim above. ❏

Finally, we prove Theorem 16.

Theorem 16 Assume $\mathcal{L} \in \{\mathcal{ALCH}, \mathcal{ALCHI}, \mathcal{ALCO}@, \mathcal{ALCHO}@, \mathcal{ALCHIO}@\}$. Then Δ-parallel interpolants exist in $\mathcal{L}$ for every $(\mathcal{T}_1, \mathcal{T}_2)$ in $\mathcal{L}$ and $\mathcal{L}$-inclusion α such that Conditions 1.-3. from Definition 11 hold and Δ contains all role and individual names in $\mathcal{T}_1, \mathcal{T}_2, \alpha$.

Proof. We provide proofs for $\mathcal{ALCH}$ and $\mathcal{ALCO}@$. The remaining proofs are similar and left to the reader. Assume that $\mathcal{T}_1, \mathcal{T}_2$ are $\mathcal{ALCH}$-TBoxes satisfying the Conditions 1.-3. for a signature Δ containing all role names in $\mathcal{T}_1$, $\mathcal{T}_2$, and α.

Assume first that α is a concept inclusion $C \sqsubseteq D$ and let

$$\mathcal{D} = \{r \sqsubseteq s \mid \mathcal{T}_1 \models r \sqsubseteq s\}.$$

Note that $\mathsf{sig}(\mathcal{D}) \subseteq \Delta$ and $\mathcal{D} = \{r \sqsubseteq s \mid \mathcal{T}_2 \models r \sqsubseteq s\}$ because $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{ALCH}$.

Let $\Box^{\leq n}\mathcal{T}$ be a shorthand for the set

$$\bigsqcap_{r_1,\dots,r_i \in \Delta, i \leq n, C \sqsubseteq D \in \mathcal{T}} \forall r_1. \cdots \forall r_i. \neg C \sqcup D.$$

One can readily show that

$$\mathcal{D} \cup \bigcup_{n<\omega} \Box^{\leq n}\mathcal{T}_1 \cup \bigcup_{i<\omega} \Box^{\leq n}\mathcal{T}_2 \models \alpha.$$

By compactness, there exists $n < \omega$ such that

$$\mathcal{D} \cup \Box^{\leq n}\mathcal{T}_1 \cup \Box^{\leq n}\mathcal{T}_2 \models \alpha$$

Hence,

$$\mathcal{D} \models (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_1} F) \sqsubseteq (((\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F) \sqcap C) \rightarrow D)$$

It follows from the analysis of the interpolation property in modal and hybrid logics in (ten Cate 2005) that there exists a $\mathcal{ALC}$-concept E_1 such that *all concept names* in $\mathsf{sig}(E_1)$ are in

$$\mathsf{sig}(\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_1} F) \cap \mathsf{sig}(((\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F) \sqcap C) \rightarrow D)$$

and the following hold:

$$\mathcal{D} \models (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_1} F) \sqsubseteq E_1,$$

$$\mathcal{D} \models E_1 \sqsubseteq (((\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F) \sqcap C) \rightarrow D).$$

We obtain

(a1) $\mathcal{T}_1 \models \top \sqsubseteq E_1$

since $\mathcal{T}_1 \models \mathcal{D}$ and because $\mathcal{T}_1 \models \top \sqsubseteq (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_1} F)$. Moreover,

(a2) $\mathsf{sig}(E_1) \setminus \Delta \subseteq \mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\alpha)$.

Next observe that a simple rewriting gives

$$\mathcal{D} \models (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F) \sqsubseteq ((E_1 \sqcap C) \rightarrow D)$$

Again by (ten Cate 2005) there exists a $\mathcal{ALC}$-concept E_2 such that all concept names in $\mathsf{sig}(E_2)$ are in

$$\mathsf{sig}(\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F) \cap \mathsf{sig}((E_1 \sqcap C) \rightarrow D)$$

such that

$$\mathcal{D} \models (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F) \sqsubseteq E_2$$

and

$$\mathcal{D} \models E_2 \sqsubseteq (E_1 \sqcap C) \rightarrow D).$$

But then

(a3) $\mathcal{T}_2 \models \top \sqsubseteq E_2$

since $\mathcal{T}_2 \models \mathcal{D}$ and because $\mathcal{T}_2 \models \top \sqsubseteq (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2} F)$. Moreover,

(a4) $\mathsf{sig}(E_2) \setminus \Delta \subseteq \mathsf{sig}(\mathcal{T}_2) \cap \mathsf{sig}(\alpha)$.

Finally, we have

(a5) $\mathcal{D} \cup \{\top \sqsubseteq E_1, \top \sqsubseteq E_2\} \models C \sqsubseteq D$.

Let $\mathcal{T}_1' = \mathcal{D} \cup \{\top \sqsubseteq E_1\}$ and $\mathcal{T}_2' = \mathcal{D} \cup \{\top \sqsubseteq E_2\}$. It follows from (a1)–(a5) that the pair $(\mathcal{T}_1', \mathcal{T}_2')$ is a Δ-parallel interpolant of $(\mathcal{T}_1, \mathcal{T}_2)$ and α in $\mathcal{ALCH}$.

Now let α be a role inclusion, say $\alpha = r \sqsubseteq s$. Assume $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \models \exists r.A \sqsubseteq \exists s.A$ for a fresh concept name A. Hence, by the result proved above, there is a Δ-parallel interpolant $(\mathcal{T}_1', \mathcal{T}_2')$ of $(\mathcal{T}_1, \mathcal{T}_2)$ and $\exists r.A \sqsubseteq \exists s.A$. We have $\mathsf{sig}(\mathcal{T}_i') \subseteq \Delta$ for $i = 1, 2$. By Δ-inseparability, $\mathcal{T}_i \models \mathcal{T}_1' \cup \mathcal{T}_2'$ for $i = 1, 2$. But then $\mathcal{T}_1 \models \exists r.A \sqsubseteq \exists s.A$. Since A is fresh, this implies $\mathcal{T}_1 \models r \sqsubseteq s$. Thus $(\{\alpha\}, \{\alpha\})$ is a Δ-parallel interpolant.

Now we consider $\mathcal{ALCO}@$. Assume that $\mathcal{T}_1, \mathcal{T}_2$ are $\mathcal{ALCO}@$-TBoxes satisfying the conditions (s1)-(s3) for a signature Δ containing all role and individual names in $\mathcal{T}_1$, $\mathcal{T}_2$, and α. Assume $\alpha = C \sqsubseteq D$.

Let N denote the set of individual names that occur in $\mathcal{T}_1, \mathcal{T}_2, \alpha$.

One can readily show that

$$\{@_a F \mid \exists n \; (F \in \Box^{\leq n}\mathcal{T}_1 \cup \Box^{\leq n}\mathcal{T}_2), a \in N\} \models \alpha.$$

Thus, by compactness, there exists $n < \omega$ with

$$\{@_a F \mid F \in \Box^{\leq n}\mathcal{T}_1 \cup \Box^{\leq n}\mathcal{T}_2, a \in N\} \models \alpha$$

Hence,

$$\models (\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_1, a \in N} @_a F) \sqsubseteq (((\bigsqcap_{F \in \Box^{\leq n}\mathcal{T}_2, a \in N} @_a F) \sqcap C) \rightarrow D)$$

It follows again from interpolation results in (ten Cate 2005) that there exists a $\mathcal{ALCO}@$-concept E_1 such that all concept names in $\mathsf{sig}(E_1)$ are in

$$\mathsf{sig}(\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_1} F)\cap \mathsf{sig}(((\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_2} F)\sqcap C)\rightarrow D)$$

and the following hold:

$$\models (\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_1, a\in N} @_a F)\sqsubseteq E_1,$$

$$\models E_1 \sqsubseteq (((\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_2, a\in N} @_a F)\sqcap C)\rightarrow D).$$

Similar to the proof above, we obtain

(a1) $\mathcal{T}_1 \models \top \sqsubseteq E_1$

(a2) $\mathsf{sig}(E_1)\setminus\Delta\subseteq \mathsf{sig}(\mathcal{T}_1)\cap\mathsf{sig}(\alpha)$.

Next observe that a simple rewriting gives

$$\models (\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_2, a\in N} @_a F)\sqsubseteq ((E_1\sqcap C)\rightarrow D)$$

By (ten Cate 2005) there exists a $\mathcal{ALCO}@$-concept E_2 such that all concept names in $\mathsf{sig}(E_2)$ are in

$$\mathsf{sig}(\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_2} F)\cap \mathsf{sig}(E_1\sqcap C)\rightarrow D)$$

such that

$$\models (\bigsqcap_{F\in\Box^{\leq n}\mathcal{T}_2, a\in N} F)\sqsubseteq E_2$$

and

$$\models E_2\sqsubseteq (E_1\sqcap C)\rightarrow D).$$

But then

(a3) $\mathcal{T}_2 \models \top\sqsubseteq E_2$

(a4) $\mathsf{sig}(E_2)\setminus\Delta\subseteq\mathsf{sig}(\mathcal{T}_2)\cap\mathsf{sig}(\alpha)$.

(a5) $\models \{\top\sqsubseteq E_1, \top\sqsubseteq E_2\}\models C\sqsubseteq D$.

Let $\mathcal{T}_1' = \{\top\sqsubseteq E_1\}$ and $\mathcal{T}_2' = \{\top\sqsubseteq E_2\}$. It follows from (a1)–(a5) that the pair $(\mathcal{T}_1', \mathcal{T}_2')$ is a Δ-parallel interpolant of $(\mathcal{T}_1,\mathcal{T}_2)$ and α in $\mathcal{ALCO}@$. ❑

Proofs for the section "Decomposition in DL-Lite"

In this section, we prove the PIP and give polynomial-time decomposition algorithms for dialects of DL-Lite. We also prove the result stated for $\mathcal{EL}$-decompositions. The algorithms presented in this section transform (in poly-time) a given TBox $\mathcal{T}$ into a fresh TBox $\mathcal{T}'$ for which the syntactic Δ-decomposition coincides with the finest Δ-decomposition of the original TBox.

Once we know that a certain DL has the PIP, we will often not state explicitly in which language we decompose a given TBox; we will also often use the fact that then the finest Δ-decompositions exists without explicitly mentioning it. It should be clear that the finest Δ-decomposition in $\mathcal{L}$ of a TBox $\mathcal{T}$ in which every axiom is Δ-indecomposable in $\mathcal{L}$ coincides with its syntactic decomposition $\mathsf{sdeco}_\Delta(\mathcal{T})$ and can, therefore, be computed in polynomial time.

Note that every concept using at most one non-Δ-symbol is $(\mathcal{T},\Delta)$-indecomposable (in every $\mathcal{T}$ and in any language). In particular, every concept name and every concept using only symbols from Δ is $(\mathcal{T},\Delta)$-indecomposable in every $\mathcal{T}$ and in any language.

Proofs for DL-Lite$_{core}$

In this subsection, we consider DL-Lite$_{core}$. We identify $\neg\neg B$ with B for any concept B. Recall that DL-Lite$_{core}$-inclusions are of the form

- $B_1\sqsubseteq B_2$, where B_1, B_2 are basic DL-Lite concepts, or
- $B_1\sqsubseteq\neg B_2$, where B_1,B_2 are basic DL-Lite concepts.

We provide a direct proof of the PIP and of the correctness of the algorithm for computing the finest Δ-decomposition of DL-Lite$_{core}$-TBoxes. Note that we could instead employ the results proved below for DL-Lite$_{horn}$.

Lemma 37 *Let $\mathcal{T}$ be a satisfiable DL-Lite$_{core}$ TBox. Then for any concept inclusion $C\sqsubseteq D$, where $C, D\in\mathsf{NB}$, we have $\mathcal{T}\models C\sqsubseteq D$ if, and only if, one of the following alternatives holds.*

1. *$\mathcal{T}\models C\sqsubseteq\bot$ and there exist $B_1,\dots,B_n\in\mathsf{NB}$ such that $B_1 = C$ and*
   - *$B_n=\bot$ or*
   - *there are $i,j\leq n$ such that $B_i=\neg B_j$,*

   and for any $i\in\{1,\dots,n-1\}$ we have
   - *$B_i\sqsubseteq B_{i+1}\in\mathcal{T}$ or*
   - *$\neg B_{i+1}\sqsubseteq\neg B_i\in\mathcal{T}$ or*
   - *$B_i=\exists r$, $B_{i+1}=\exists r^-$, for some role name r, or*
   - *$B_i=\exists r^-$, $B_{i+1}=\exists r$, for some role name r.*
2. *$\mathcal{T}\models\top\sqsubseteq D$ and there exist $B_1,\dots,B_n\in\mathsf{NB}$ such that $B_1=\top$, $B_n=D$ and for any $i\in\{1,\dots,n-1\}$ we have*
   - *$B_i\sqsubseteq B_{i+1}\in\mathcal{T}$ or*
   - *$\neg B_{i+1}\sqsubseteq\neg B_i\in\mathcal{T}$.*
3. *$\mathcal{T}\not\models C\sqsubseteq\bot$, $\mathcal{T}\not\models\top\sqsubseteq D$ and there exist $B_1,\dots,B_n\in\mathsf{NB}$ such that $B_1=C$, $B_n=D$ and for any $i\in\{1,\dots,n-1\}$ we have*
   - *$B_i\sqsubseteq B_{i+1}\in\mathcal{T}$ or*
   - *$\neg B_{i+1}\sqsubseteq\neg B_i\in\mathcal{T}$.*

Proof. We prove that if there do not exist $B_1,\dots,B_n$ satisfying the conditions in Point 1, then $\mathcal{T}\not\models C\sqsubseteq\bot$. Corresponding claims can be proved in the same way for Point 2 and 3 and are left to the reader.

So, assume $C\sqsubseteq D$ is given and that

$(*)$ there do not exist $B_1,\dots,B_n$ satisfying the conditions in Point 1.

To prove that $\mathcal{T}\not\models C\sqsubseteq\bot$ we construct a model $\mathcal{I}^*$ of $\mathcal{T}$ satisfying C.

For B a basic concept or its negation define

$$\mathscr{C}(B)=\left\{B' \;\middle|\; \begin{array}{l} B'\in\mathsf{NB}, \exists B_1',\dots,B_l' : B_1'=B, B_l'=B' \\ \forall i<l : B_i'\sqsubseteq B_{i+1}'\in\mathcal{T} \text{ or } \neg B_{i+1}'\sqsubseteq\neg B_i'\in\mathcal{T}\end{array}\right\}$$

We build $\mathcal{I}^*$ by constructing interpretations interpretation $\mathcal{I}^i = (\Delta^{\mathcal{I}_i}, \cdot^{\mathcal{I}^i})$, $i \geq 0$ and then define $\mathcal{I}^*$ as

$$\begin{array}{rcl} \Delta^{\mathcal{I}^*} & = & \bigcup_{i\geq 0} \Delta^{\mathcal{I}_i} \\ A^{\mathcal{I}^*} & = & \bigcup_{i\geq 0} A^{\mathcal{I}_i}, \text{ for all } A \in \mathsf{N_C} \text{ and} \\ r^{\mathcal{I}^*} & = & \bigcup_{i\geq 0} r^{\mathcal{I}_i}, \text{ for all } r \in \mathsf{N_R}. \end{array}$$

Let

$$\begin{array}{rcl} \Delta^{\mathcal{I}_0} & = & \{d_0\}, \\ \mathscr{C}(d_0) & = & \mathscr{C}(C), \\ A^{\mathcal{I}_0} & = & \{d_0 \mid A \in \mathscr{C}(C)\}, \text{ for all } A \in \mathsf{N_C}, \text{ and} \\ r^{\mathcal{I}_0} & = & \emptyset, \text{ for all } r \in \mathsf{N_R}. \end{array}$$

For uniformity of the notation, we define $\Delta^{-1} = \emptyset$. For $i \geq 0$ we set

$$\Delta^{\mathcal{I}_{i+1}} = \Delta^{\mathcal{I}_i} \cup \bigcup_{\substack{d_i \in \Delta^{\mathcal{I}_i} \setminus \Delta^{\mathcal{I}_{i-1}}, \\ \exists r \in \mathscr{C}(d_i)}} \{d_{i+1}^{\exists r^-}\} \cup \bigcup_{\substack{d_i \in \Delta^{\mathcal{I}_i} \setminus \Delta^{\mathcal{I}_{i-1}}, \\ \exists r^- \in \mathscr{C}(d_i)}} \{d_{i+1}^{\exists r}\},$$

where $d_{i+1}^{\exists r^-}$ and $d_{i+1}^{\exists r}$ represent new elements added to the domain $\Delta^{\mathcal{I}_i}$ when we are to satisfy $\exists r$ and $\exists r^-$;

$$\mathscr{C}(d_{i+1}^{\exists r}) = \mathscr{C}(\exists r); \quad \mathscr{C}(d_{i+1}^{\exists r^-}) = \mathscr{C}(\exists r^-);$$

$$r^{\mathcal{I}_{i+1}} = r^{\mathcal{I}_i} \cup \bigcup_{\substack{d_i \in \Delta^{\mathcal{I}_i} \setminus \Delta^{\mathcal{I}_{i-1}}, \\ \exists r \in \mathscr{C}(d_i)}} \{(d_i, d_{i+1}^{\exists r^-})\} \cup \bigcup_{\substack{d_i \in \Delta^{\mathcal{I}_i} \setminus \Delta^{\mathcal{I}_{i-1}}, \\ \exists r^- \in \mathscr{C}(d_i)}} \{(d_{i+1}^{\exists r}, d_i)\},$$

for all $r \in \mathsf{N_R}$;

$$A^{\mathcal{I}_{i+1}} = A^{\mathcal{I}_i} \cup \{d_{i+1}^{\exists r} \mid A \in \mathscr{C}(\exists r)\} \cup \{d_{i+1}^{\exists r^-} \mid A \in \mathscr{C}(\exists r^-)\},$$

for all $A \in \mathsf{N_C}$. It follows immediately from $(*)$ that $\mathscr{C}(d_0)$ does not contain $\bot$ nor any $\{B, \neg B\}$. Then, using again $(*)$ it can be shown that no $\mathscr{C}(\exists r)$ (and no $\mathscr{C}(\exists r^-)$) used in the definition of any $A^{\mathcal{I}_{i+1}}$ contains $\bot$ nor any $\{B, \neg B\}$. It is now straightforward to prove that $\mathcal{I}^*$ is a model of $\mathcal{T}$ and $d_0 \in C^{\mathcal{I}^*}$, as required. ❏

Corollary 38 *DL-Lite$_{core}$ has the PIP.*

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be DL-Lite$_{\text{core}}$ TBoxes, $C \sqsubseteq D$ a concept inclusion, and Δ a signature such that $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$, and $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. DL-Lite$_{\text{core}}$.

We consider the case $\mathsf{sig}(C) \subseteq \Sigma_1$, $\mathsf{sig}(D) \subseteq \Sigma_2$, $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models C \sqsubseteq \bot$, $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models \top \sqsubseteq D$ (the remaining cases are similar and left to the reader). Then, by Lemma 37, there exist $B_1, \dots, B_n \in \mathsf{NB}$ such that $B_1 = C$, $B_n = D$ and for any $i \in \{1, \dots, n-1\}$ we have

- $B_i \sqsubseteq B_{i+1} \in \mathcal{T}_1 \cup \mathcal{T}_2$ or
- $\neg B_{i+1} \sqsubseteq \neg B_i \in \mathcal{T}_1 \cup \mathcal{T}_2$.

Since $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, there must exist i, j such that $B_i \in \Delta$, $B_j \in \Delta$, $B_1, \dots, B_{i-1} \in \Sigma_1$, $B_{j+1}, \dots, B_n \in \Sigma_2$. Using the condition that $\mathcal{T}_1$, $\mathcal{T}_2$ are Δ-inseparable, one can now easily prove $\mathcal{T}_k \models B_i \sqsubseteq B_j$ for $k = 1, 2$. Thus, $\mathcal{T}_1' = \{C \sqsubseteq B_j\}$ and $\mathcal{T}_2' = \{B_j \sqsubseteq D\}$ form a Δ-parallel interpolant of $(\mathcal{T}_1, \mathcal{T}_2)$ and $C \sqsubseteq D$. ❏

Lemma 39 *Let $\mathcal{T}$ be a DL-Lite$_{core}$-TBox. Let $B_1 \sqsubseteq B_2$ be a DL-Lite$_{core}$ inclusion with $\mathcal{T} \models B_1 \sqsubseteq B_2$. If $B_1 \sqsubseteq B_2$ is not $(\mathcal{T}, \Delta)$-indecomposable, then there exists $D \in \mathsf{NB}$ such that $\mathsf{sig}(D) \subseteq \Delta$ and $\mathcal{T} \models B_1 \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq B_2$.*

Proof. Follows from Lemma 37. ❏

From this lemma and the PIP we immediately obtain the following theorem which implies that our algorithm for computing the finest Δ-decomposition of DL-Lite$_{core}$-TBoxes is correct.

Theorem 19 *DL-Lite$_{core}$ has the PIP. For DL-Lite$_{core}$-TBoxes $\mathcal{T}$ and signatures Δ, one can compute in polynomial time a realization in DL-Lite$_{core}$ of the finest Δ-decomposition $\mathcal{T}$.*

Proofs for DL-Lite$_{\text{horn}}$

In what follows we prove that given a DL-Lite$_{\text{horn}}$ TBox the algorithm in Fig. 2 outputs a TBox in which every axiom is Δ-indecomposable. We consider the *propositional* case first, that is, we assume that expressions of the form $\exists r$ and $\exists r^-$ do not occur in the given TBox. In this case, a concept is satisfiable if, and only if, it is satisfiable in a one-element model. Then for any interpretation $\mathcal{I}$ with $D^{\mathcal{I}} = \{d\}$ we denote $d \in C^{\mathcal{I}}$ by $\mathcal{I} \models C$, for any propositional concept C.

We will use the following naming convention. Fix a TBox $\mathcal{T}$ and mutually disjoint sets $\Delta, \Sigma_1, \Sigma_2$ such that $\Delta \cup \Sigma_1 \cup \Sigma_2 = \Sigma$ for $\Sigma = \mathsf{sig}(\mathcal{T})$. Consider fresh signatures $\Sigma' = \{x' \mid x \in \Sigma\}$ and $\Sigma^\dagger = \{x^\dagger \mid x \in \Sigma\}$. Let C ($C \sqsubseteq B$ or $\mathcal{T}$) be a Σ-concept (Σ-concept inclusion or Σ-TBox, resp.). By C_{Σ_1} ($(C \sqsubseteq B)_{\Sigma_1}$ or $\mathcal{T}_{\Sigma_1}$, resp.) we denote a concept (concept inclusion, TBox, resp.) obtained from C ($C \sqsubseteq B$ or $\mathcal{T}$) by replacing every occurrence of a symbol $x \in \Sigma_2$ with x'. By C_{Σ_2} ($(C \sqsubseteq B)_{\Sigma_2}$ or $\mathcal{T}_{\Sigma_2}$, resp.) we denote a concept (concept inclusion, TBox, resp.) obtained from C ($C \sqsubseteq B$ or $\mathcal{T}$) by replacing every occurrence of a symbol $x \in \Sigma_1$ with $x^\dagger$.

Lemma 40 *Let $\mathcal{T}$ be a propositional TBox and Δ, Σ_1, and Σ_2 mutually disjoint signatures such that $\mathsf{sig}(\mathcal{T}) = \Delta \cup \Sigma_1 \cup \Sigma_2$. For any $B \in \mathsf{sig}(\mathcal{T}_{\Sigma_1}) \cup \{\bot\}$, if*

$$\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2} \models C \sqsubseteq B$$

then

$$\mathcal{T}_{\Sigma_1} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_1}, \Delta \cup (\Sigma_1 \cap \mathsf{sig}(C))}(C_{\Sigma_1}) \sqsubseteq B$$

Proof. Suppose that C is of the form $C_1 \sqcap C_2 \sqcap C_3$, where $\mathsf{sig}(C_1) \subseteq \Delta$, $\mathsf{sig}(C_2) \subseteq \Sigma_1$, and $\mathsf{sig}(C_3) \subseteq \Sigma_2$.

If $\mathcal{T}_{\Sigma_1} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_1},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C_{\Sigma_1}) \sqsubseteq \bot$, then the lemma is trivial. Thus, we can assume that there exists a model of $\mathcal{T}_{\Sigma_1}$ satisfying $\mathsf{Cons}_{\mathcal{T}_{\Sigma_1},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C_{\Sigma_1})$.

Take the (uniquely determined) model $\mathcal{I}_1$ of $\mathcal{T}_{\Sigma_1}$ such that for any concept A we have $\mathcal{I}_1 \models A$ if, and only if

$$\mathcal{T}_{\Sigma_1} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_1},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C_{\Sigma_1}) \sqsubseteq A.$$

Notice that $\mathcal{I}_1 \models C_1 \sqcap C_2$.

Observe that, for all $A \in \Delta \cup \{\bot\}$, the following conditions are equivalent:

- $\mathcal{T}_{\Sigma_1} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_1},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C_{\Sigma_1}) \sqsubseteq A$,
- $\mathcal{T}_{\Sigma_2} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_2},\Delta\cup(\Sigma_2\cap\mathsf{sig}(C))}(C_{\Sigma_2}) \sqsubseteq A$.

Thus, we also have the (uniquely determined) model $\mathcal{I}_2$ of $\mathcal{T}_{\Sigma_2}$ such that any concept A we have $\mathcal{I}_2 \models A$ if, and only if

$$\mathcal{T}_{\Sigma_2} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_2},\Delta\cup(\Sigma_2\cap\mathsf{sig}(C))}(C_{\Sigma_2}) \sqsubseteq A.$$

We have $\mathcal{I}_2 \models C_3$. We have

- for any concept $A \in \Delta$: $\mathcal{I}_1 \models A$ if, and only if $\mathcal{I}_2 \models A$;
- for any $A \in \mathsf{sig}(\mathcal{T}_{\Sigma_2}) \setminus \Delta$: $\mathcal{I}_1 \not\models A$;
- for any $A \in \mathsf{sig}(\mathcal{T}_{\Sigma_1}) \setminus \Delta$: $\mathcal{I}_2 \not\models A$.

Thus, $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ is a model for $\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2}$ and $\mathcal{I} \models C$.

Now assume that $\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2} \models C \sqsubseteq B$. Then $\mathcal{I} \models B$. Hence $\mathcal{I}_1 \models B$ and we obtain $\mathcal{T}_{\Sigma_1} \models \mathsf{Cons}_{\mathcal{T}_{\Sigma_1},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C_{\Sigma_1}) \sqsubseteq B$, as required. ❑

Theorem 22 *For any propositional DL-Lite$_{horn}$ TBox $\mathcal{T}$, the algorithm in Figure 2 runs in poly-time and outputs a TBox $\mathcal{T}'$ in which every CI is Δ-indecomposable w.r.t. $\mathcal{T}$. Thus, $\mathsf{sdeco}_\Delta(\mathcal{T}')$ coincides with the finest Δ-decomposition of $\mathcal{T}$.*

Proof. First, we show that the algorithm in Fig. 2 terminates and prove the complexity bound. Let N be the number of axioms in $\mathcal{T}$ and $L = \max\{|\mathsf{sig}(C) \setminus \Delta| \mid C \sqsubseteq B \in \mathcal{T}\}$. Notice that no transformation rule is applicable to $\alpha_1 = \mathsf{Cons}_{\mathcal{T},\Delta}(C) \sqsubseteq B$ introduced in line 8 of the algorithm (since $|\mathsf{sig}(\alpha_1) \setminus \Delta| = 1$). Notice further that the condition in Line 5 does not hold for $\alpha_2 = C \sqsubseteq B'$ introduced in Line 8 nor for any rewritings of α_2 (since the rewriting rule in Line 12 does not change the right-hand size of an axiom and $\mathsf{sig}(B') \subseteq \Delta$). Therefore, Line 12 can execute at most $|\Delta| \cdot N \cdot L^2$ times; and line 8 can execute at most N times.

We prove now that if a TBox $\mathcal{T}$ has properties (Red), (Def), and (Int), then every CI in $\mathcal{T}$ is Δ-indecomposable w.r.t. $\mathcal{T}$.

Suppose that some $\alpha = C \sqsubseteq B \in \mathcal{T}$ is not Δ-indecomposable then either there exist a signature Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$ such that $\mathsf{sig}(C) \cap \Sigma_i \neq \emptyset$ for $i = 1, 2$ or there exist a signature Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$ such that $\mathsf{sig}(C) \cap \Sigma_2 \neq \emptyset$, $\mathsf{sig}(C) \subseteq \Sigma_2 \cup \Delta$, and $B \in \Sigma_1$. We show that in both cases these assumptions lead to a contradiction.

Consider the first case. Suppose that there exists $i \in \{1, 2\}$ such that

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \models C \sqsubseteq C_{\Sigma_i}$$

Then, by Lemma 40, since $\mathsf{sig}(C_{\Sigma_i}) \subseteq \mathsf{sig}(\mathcal{T}_i)$,

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_i} \models \mathsf{Cons}_{(\mathcal{T}\setminus\{\alpha\})_{\Sigma_i},\Delta\cup(\Sigma_i\cap\mathsf{sig}(C)))}(C_{\Sigma_i}) \sqsubseteq C_{\Sigma_i}.$$

After renaming the signatures appropriately, we get

$$\mathcal{T} \setminus \{\alpha\} \models \mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},\Delta\cup(\Sigma_i\cap\mathsf{sig}(C)))}(C) \sqsubseteq C,$$

contradicting (Def). Therefore,

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \not\models (C \sqsubseteq C_{\Sigma_i})$$

for $i = 1, 2$. Consider a model $\mathcal{I}$ of $(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2}$ such that we have for any concept name A: $\mathcal{I} \models A$ if, and only if $(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \models C \sqsubseteq A$. In particular, $\mathcal{I} \models C$. Then $\mathcal{I} \not\models C_{\Sigma_i}$ for $i = 1, 2$. Thus, $\mathcal{I} \models C_{\Sigma_i} \sqsubseteq B_{\Sigma_i}$, and so $\mathcal{I} \models \mathcal{T}_{\Sigma_i}$. Since Σ_1, Σ_2 is a decomposition of $\mathcal{T}$ and $\mathcal{I} \models \alpha$, we have

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \models \alpha$$

contradicting (Red), since $|\mathsf{sig}(\alpha) \setminus \Delta \geq 2|$.

Consider now the second case. By Lemma 40 (using renaming), $\mathcal{T} \models \mathsf{Cons}_{\mathcal{T},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C) \sqsubseteq B$. Since $\Sigma_1 \cap \mathsf{sig}(C) = \emptyset$, $\mathsf{Cons}_{\mathcal{T},\Delta\cup(\Sigma_1\cap\mathsf{sig}(C))}(C) = \mathsf{Cons}_{\mathcal{T},\Delta}(C)$, therefore, $\mathcal{T} \models \mathsf{Cons}_{\mathcal{T},\Delta}(C) \sqsubseteq B$, contradicting (Int). ❑

Lemma 41 *Let $\mathcal{T}$ be a DL-Lite$_{horn}$ TBox and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$ a signature such that*

- *if $\mathcal{T} \models \exists r \sqsubseteq \bot$ for some role r then $\exists r \sqsubseteq \bot \in \mathcal{T}$;*
- *every axiom in $\mathcal{T}^P$ is Δ^P-indecomposable.*

Then every axiom in $\mathcal{T}$ is Δ-indecomposable.

Proof. Suppose there exists an $\alpha \in \mathcal{T}$, which is not Δ-indecomposable. Then there exists a Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$, such that $\Sigma_i \cap \mathsf{sig}(\alpha) \neq \emptyset$ for $i = 1, 2$. Let $\mathcal{T}_1$, $\mathcal{T}_2$ realise Σ_1, Σ_2. Then, by Lemma **??**, $\mathcal{T}_1^P \cup \mathcal{T}_2^P \equiv \mathcal{T}^P$. Thus, $\mathcal{T}_1^P, \mathcal{T}_2^P$ realize the Δ^P-decomposition $\mathsf{sig}(\mathcal{T}_1^P) \setminus \Delta$, $\mathsf{sig}(\mathcal{T}_2^P) \setminus \Delta$ of $\mathcal{T}^P$. This shows that α^P is not Δ^P-indecomposable, and so we have derived a contradiction. ❑

Notice that the converse does not hold. Consider $\mathcal{T} = \{\exists r \sqcap \exists r^- \sqsubseteq A, \exists r \sqsubseteq B, B \sqsubseteq \exists r\}$ and $\Delta = \{B\}$. Then every axiom in $\mathcal{T}$ is Δ-indecomposable but its propositional counterpart $\mathcal{T}^P = \{P_{\exists r} \sqcap P_{\exists r^-} \sqsubseteq A, P_{\exists r} \sqsubseteq B, B \sqsubseteq P_{\exists r^-}\}$ can be decomposed as $\{B \sqcap P_{\exists r^-} \sqsubseteq A, P_{\exists r} \sqsubseteq B, B \sqsubseteq P_{\exists r^-}\}$.

Theorem 24 *The algorithm $\mathsf{Rewrite}_{\textit{DL-Lite}_{horn}}$ given in Fig. 3 transforms a given DL-Lite$_{horn}$ TBox into an equivalent DL-Lite$_{horn}$ TBox in which every axiom is Δ-indecomposable.*

Proof. Clearly, $(\mathcal{T}')^P = \mathcal{T}_{\mathrm{Res}}^P$ so, by Lemma 41, every axiom in $\mathcal{T}'$ is Δ-indecomposable. ❑

Proofs for DL-Lite$^{\mathcal{H}}_{\text{horn}}$

To prove the PIP for DL-Lite$^{\mathcal{H}}_{\text{horn}}$ we prove first prove that reasoning in DL-Lite$^{\mathcal{H}}_{\text{horn}}$ can be reduced to reasoning in DL-Lite$_{\text{horn}}$.

Lemma 25 *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{horn}$ TBox and Δ a signature. Let $\mathcal{T}^0$ be the set of CIs in $\mathcal{T}$ and set*

$$\mathcal{T}' = \mathcal{T}^0 \cup \{\exists r \sqsubseteq \exists s, \exists r^- \sqsubseteq \exists s^- \mid r \sqsubseteq s \in \mathcal{T}\}.$$

Then $\mathcal{T} \models \alpha$ if, and only if, $\mathcal{T}' \models \alpha$ for all CIs α in DL-Lite$_{horn}$.

Proof. The "if" direction is obvious. For the "only if" direction we show that given an arbitrary model $\mathcal{I}'$ of $\mathcal{T}'$ one can construct a model $\mathcal{I}$ of $\mathcal{T}$ such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$ and for any $d \in \Delta^{\mathcal{I}}$ and any DL-Lite$_{\text{horn}}$ concept D we have $d \in D^{\mathcal{I}}$ if, and only if, $d \in D^{\mathcal{I}'}$. The required $\mathcal{I}$ differs from $\mathcal{I}'$ only in the way how roles are interpreted, namely, for any role name s:

$$s^{\mathcal{I}} = s^{\mathcal{I}'} \cup \bigcup_{r \sqsubseteq^*_{\mathcal{T}} s} r^{\mathcal{I}'}, \qquad (*)$$

where $r \sqsubseteq^*_{\mathcal{T}} s$ denotes there exist $t_1, \dots, t_n$ such that $r = t_1$, $s = t_n$ and $t_i \sqsubseteq t_{i+1} \in \mathcal{T}$ or $t_i^- \sqsubseteq t_{i+1}^- \in \mathcal{T}$. That is, whenever $(u, v) \in r^{\mathcal{I}}$ and $r \sqsubseteq s \in \mathcal{T}$ we also have $(u, v) \in s^{\mathcal{I}}$. It should be clear that for any $d \in \Delta^{\mathcal{I}}$ and DL-Lite$_{\text{horn}}$ concept C we have $d \in C^{\mathcal{I}'}$ implies $d \in C^{\mathcal{I}}$. We prove the inverse by induction on the structure of C. Since the interpretation of concept names in $\mathcal{I}$ coincides with $\mathcal{I}'$, we have $d \in A^{\mathcal{I}}$ implies $d \in A^{\mathcal{I}'}$. Let C be of the form $\exists s$ and $d \in (\exists s)^{\mathcal{I}}$. Then there exists $d' \in \Delta^{\mathcal{I}}$ such that $(d, d') \in s^{\mathcal{I}}$, that is either $(d, d') \in s^{\mathcal{I}'}$ or $(d, d') \in r^{\mathcal{I}'}$ for some role r such that $r \sqsubseteq^*_{\mathcal{T}} s$. In the latter case, $d \in (\exists r)^{\mathcal{I}'}$ and, since $\mathcal{T}' \models \exists r \sqsubseteq \exists s$, $d \in (\exists s)^{\mathcal{I}'}$. The case of a conjunctive C is trivial. Every role inclusion of the form $r \sqsubseteq s \in \mathcal{T}$ is satisfied in $\mathcal{I}$ because of $(*)$. Thus $\mathcal{I}$ is a model of $\mathcal{T}$. ❏

Corollary 42 *DL-Lite$^{\mathcal{H}}_{horn}$ has the PIP.*

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be DL-Lite$^{\mathcal{H}}_{\text{horn}}$ TBoxes, α a concept or role inclusion, and Δ a signature such that $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, $\mathcal{T}_1 \cup \mathcal{T}_2 \models \alpha$, and $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. DL-Lite$^{\mathcal{H}}_{\text{horn}}$. Consider cases.

Suppose that $\alpha = C \sqsubseteq B$ is a concept inclusion. Notice that C and B are DL-Lite$_{\text{horn}}$ concepts, so, by Lemma 25, $\mathcal{T}_1^0 \cup \mathcal{T}_2^0 \cup \{\exists r \sqsubseteq \exists s, \exists r^- \sqsubseteq \exists s^- \mid r \sqsubseteq s \in (\mathcal{T}_1 \cup \mathcal{T}_2)\} \models C \sqsubseteq B$, where $\mathcal{T}_i^0$ is the set of concept inclusion in $\mathcal{T}_i$ for $i = 1, 2$. Let $\mathcal{T}_i' = \mathcal{T}_i^0 \cup \{\exists r \sqsubseteq \exists s, \exists r^- \sqsubseteq \exists s^- \mid r \sqsubseteq s \in \mathcal{T}_i\}$. Clearly, $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, $\mathcal{T}_1' \cup \mathcal{T}_2' \models C \sqsubseteq B$ and $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are Δ-inseparable w.r.t. DL-Lite$_{\text{horn}}$. Since DL-Lite$_{\text{horn}}$ has the PIP there exist DL-Lite$_{\text{horn}}$ TBoxes $\mathcal{T}_1''$ and $\mathcal{T}_2''$ such that $(\mathcal{T}_1'', \mathcal{T}_2'')$ is a Δ-parallel interpolant of $(\mathcal{T}_1', \mathcal{T}_2')$ and α. But then $\mathcal{T}_1 \models \mathcal{T}_1''$ and $\mathcal{T}_2 \models \mathcal{T}_2''$ and so $(\mathcal{T}_1'', \mathcal{T}_2'')$ is a Δ-parallel interpolant of $(\mathcal{T}_1, \mathcal{T}_2)$ and α, as required.

Suppose now that $\alpha = r \sqsubseteq s$ is a role inclusion such that, w.l.o.g., $r \in \mathsf{sig}(\mathcal{T}_1)$ and $s \in \mathsf{sig}(\mathcal{T}_2)$. Then either $\mathcal{T}_1 \cup \mathcal{T}_2 \models \exists r \sqsubseteq \bot$ or there exist roles $t_1, \dots, t_n \in \mathsf{sig}(\mathcal{T}_1) \cup \mathsf{sig}(\mathcal{T}_2)$ such that $r = t_1$, $t_n = s$ and $t_i \sqsubseteq t_{i+1} \in \mathcal{T}_1 \cup \mathcal{T}_2$. In the former case the role inclusion α follows from the DL-Lite$_{\text{horn}}$ concept inclusion $\exists r \sqsubseteq \bot$ so this case can be reduced to the one considered above. In the latter case, since $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$, and $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable, there exists $t \in \Delta$ such that $\mathcal{T}_1 \models r \sqsubseteq t$ and $\mathcal{T}_2 \models t \sqsubseteq s$. Let $\mathcal{T}_1' = \{r \sqsubseteq t\}$ and $\mathcal{T}_2' = \{t \sqsubseteq s\}$. Then $(\mathcal{T}_1', \mathcal{T}_2')$ is a Δ-parallel interpolant of $(\mathcal{T}_1, \mathcal{T}_2)$ and α. ❏

Lemma 43 *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{horn}$ TBox $\mathcal{T}$ and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$. Let $\mathcal{T}^0$ be the set of concept inclusions in $\mathcal{T}$. Suppose that for any role r if $\mathcal{T} \models \exists r \sqsubseteq \bot$ then $\mathcal{T}^0 \models \exists r \sqsubseteq \bot$. Let $\mathcal{T}^1$ be the set of* role *inclusions in $\mathcal{T}$, $r \sqsubseteq s \in \mathcal{T}_1$, and assume*

- *there does not exist a role $t \in \Delta$ such that $\mathcal{T}^1 \models r \sqsubseteq t$ and $\mathcal{T}^1 \models t \sqsubseteq s$;*
- *$\mathcal{T}^0 \not\models \exists r \sqsubseteq \bot$.*

Then $r \sqsubseteq s \in \mathcal{T}$ is $(\mathcal{T}, \Delta)$-indecomposable.

Proof. Assume that $\mathcal{T}$ and $r \sqsubseteq s$ satisfy the conditions of the lemma and there exists a Δ-decomposition $\Sigma_1, \dots, \Sigma_n$ of $\mathcal{T}$ such that $r \in \Sigma_i$ and $s \in \Sigma_j$ for $i \neq j$. Let $\mathcal{T}_1, \dots, \mathcal{T}_n$ realise $\Sigma_1, \dots, \Sigma_n$. Notice that $\mathcal{T} \models r \sqsubseteq s$ if, and only if, $\mathcal{T} \models \exists r \sqsubseteq \bot$ or there exist roles $t_1, \dots, t_n \in \Sigma_1 \cup \dots \cup \Sigma_n$ such that $r = t_1$, $t_n = s$ and $t_i \sqsubseteq t_{i+1} \in \mathcal{T}$. Notice that, under the conditions of the lemma, $\mathcal{T} \not\models \exists r \sqsubseteq \bot$ and $t_i \notin \Delta$ for $i = 1, \dots, n$. Then there must exists $i \in \{1, \dots, n\}$ and $k \neq l$ such that $t_i \in \Sigma_k$ and $t_i \in \Sigma_l$ contradicting $\Sigma_1, \dots, \Sigma_n$ being a Δ-decomposition of $\mathcal{T}$. ❏

Theorem 26 *The algorithm in* $\textsf{Rewrite}_{\textit{DL-Lite}^{\mathcal{H}}_{horn}}$ *given in Fig. 4 transforms a given DL-Lite$^{\mathcal{H}}_{horn}$ TBox into an equivalent DL-Lite$^{\mathcal{H}}_{horn}$ TBox in which every axiom is Δ-indecomposable.*

Proof. By Lemmas 43 and 25. ❏

Proofs for the section "Decomposition in $\mathcal{EL}$"

For the proofs in this section, we will frequently use the following result from (Lutz and Wolter 2010).

Lemma 44 *Let $\mathcal{T}$ be a $\mathcal{EL}$-TBox and C, D be $\mathcal{EL}$-concepts. Suppose $\mathcal{T} \models C \sqsubseteq \exists r.D$. Then one of the following holds:*

- *there is a conjunct $\exists r.C'$ of C such that $\mathcal{T} \models C' \sqsubseteq D$;*
- *there is a $\exists r.C' \in sub(\mathcal{T})$ such that $\mathcal{T} \models C \sqsubseteq \exists r.C'$ and $\mathcal{T} \models C' \sqsubseteq D$.*

We also require the following interpolation result that can be proved by close inspection of the proof of Theorem 36.

Theorem 45 *Let $\mathcal{T}_1, \mathcal{T}_2$ be $\mathcal{EL}$-TBoxes with $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Delta$. Assume $\mathcal{T}_1 \cup \mathcal{T}_2 \models C \sqsubseteq D$ for $\mathcal{EL}$-concepts C, D such that $\mathsf{sig}(C) \subseteq \mathsf{sig}(\mathcal{T}_1)$ and $\mathsf{sig}(D) \subseteq \mathsf{sig}(\mathcal{T}_2)$. Then there exists a $\mathcal{EL}$-concept G with $\mathsf{sig}(G) \subseteq \Delta$ such that $\mathcal{T}_1 \models C \sqsubseteq G$ and $\mathcal{T}_2 \models G \sqsubseteq D$.*

We have seen that in $\mathcal{EL}$ finest Δ-decompositions do not always have realizations of polynomial size. Thus, instead of computing the finest realization itself, we will construct a surrogate containing additional concept names as well as additional non-$\mathcal{EL}$ constructors. When introducing new concept names, we will ensure that we obtain safe variants:

Definition 46 (Safe variants) Let $\mathcal{T}$ be a TBox. A TBox $\mathcal{T}'$ is called a *safe variant* of $\mathcal{T}$ for Δ if there exists a set $\mathcal{C}$ of concepts C with $\mathsf{sig}(C) \subseteq \mathsf{sig}(\mathcal{T})$ such that each $C \in \mathcal{C}$ is $\mathcal{T}$-equivalent to a $(\mathcal{T}, \Delta)$-indecomposable concept C' (i.e., $\mathcal{T} \models C \equiv C'$) and

$$\mathcal{T}' \equiv \mathcal{T} \cup \{A_C \equiv C \mid C \in \mathcal{C}\},$$

where the A_C, $C \in \mathcal{C}$ are concept names that do not occur in $\mathcal{T}$. We call such an A_C a surrogate of C.

Lemma 47 *Let $\mathcal{T}' \equiv \mathcal{T} \cup \{A_C \equiv C \mid C \in \mathcal{C}\}$ be a safe variant of $\mathcal{T}$ for Δ and let Σ_i, $1 \leq i \leq n$, form a partition of $\mathsf{sig}(\mathcal{T}) \setminus \Delta$. The following conditions are equivalent:*

- *$\Sigma_1, \ldots, \Sigma_n$ is a signature Δ-decomposition of $\mathcal{T}$;*
- *there exist mutually disjoint subsets $\Pi_1, \ldots, \Pi_n, \Pi_{n+1}, \ldots, \Pi_m$ of $\{A_C \mid C \in \mathcal{C}\}$ such that $\Pi_{n+1}, \ldots, \Pi_m$ are non-empty and $\bigcup_{1 \leq i \leq m} \Pi_i = \{A_C \mid C \in \mathcal{C}\}$ such that $\Sigma_1 \cup \Pi_1, \ldots, \Sigma_n \cup \Pi_n, \Pi_{n+1}, \ldots, \Pi_m$ is a signature Δ-decomposition of $\mathcal{T}'$.*

It follows that if we are able to compute in polytime a safe variant $\mathcal{T}'$ of $\mathcal{T}$ for Δ and its finest Δ-decomposition, then we obtain, again in polytime, the finest Δ-decomposition of $\mathcal{T}$.

Consider the following properties of TBoxes:

(R1) If $C \sqsubseteq D \in \mathcal{T}$, then D is a concept name or of the form $\exists r.A$, where A is a concept name.

(R2) If $C \sqsubseteq \exists r.A \in \mathcal{T}$ then there does not exist a top-level conjunct $\exists r.C'$ of C with $\mathcal{T} \models C' \sqsubseteq A$.

(R3) If $C \sqsubseteq \exists r.A \in \mathcal{T}$, then for every concept D with $\mathcal{T} \models C \sqsubseteq \exists r.D$ and $\mathcal{T} \models D \sqsubseteq A$ we have $\mathcal{T} \models A \sqsubseteq D$.

We also require the following property that depends on $\Delta \subseteq \mathsf{sig}(\mathcal{T})$:

(R4) If $C \sqsubseteq \exists r.A \in \mathcal{T}$, then $\exists r.A$ is $(\mathcal{T}, \Delta)$-indecomposable.

Lemma 48 *Let $\mathcal{T}$ be a TBox and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$. Then one can construct a safe variant $\mathcal{T}'$ of $\mathcal{T}$ satisfying (R1)–(R4) in polynomial time.*

Proof. Assume $\mathcal{T}$ and Δ are given. Let $\mathcal{T}_0 := \mathcal{T}$. We apply the following rewrite rule exhaustively to $\mathcal{T}_0$:

(RR0) If $C \sqsubseteq C_1 \sqcap C_2 \in \mathcal{T}_0$, then remove it and add instead $C \sqsubseteq C_1, C \sqsubseteq C_2$ to $\mathcal{T}_0$.

(RR1) If $C \sqsubseteq \exists r.D \in \mathcal{T}_0$ and $\exists r.C'$ is a top-level conjunct of C such that $\mathcal{T}_0 \models C' \sqsubseteq D$, then remove it and add instead $C' \sqsubseteq D$ to $\mathcal{T}_0$.

(RR2) If $C \sqsubseteq \exists r.D \in \mathcal{T}_0$ and (RR1) is not applicable, then compute the set $\mathcal{D}$ of all concepts D_0 that occur in $\mathsf{sub}(\mathcal{T}_0)$ and such that

- $\mathcal{T}_0 \models C \sqsubseteq \exists r.D_0$;
- $\mathcal{T}_0 \models D_0 \sqsubseteq D$,
- there does not exist D' in $\mathsf{sub}(\mathcal{T}_0)$ with $\mathcal{T}_0 \models C \sqsubseteq \exists r.D'$ and $\mathcal{T}_0 \models D' \sqsubseteq D_0$ but $\mathcal{T}_0 \not\models D_0 \sqsubseteq D'$.

Replace $C \sqsubseteq \exists r.D \in \mathcal{T}_0$ by

$$\{A_{D_0} \sqsubseteq D_0, D_0 \sqsubseteq A_{D_0}, C \sqsubseteq \exists r.A_{D_0}, A_{D_0} \sqsubseteq D \mid D_0 \in \mathcal{D}\},$$

where the A_{D_0} are fresh concept names.

It remains to show that $\mathcal{T}_0$ has the properties (R1)-(R4). This follows immediately from the following

Claim. Let $\mathcal{T} \models C \sqsubseteq \exists r.D$ and assume (RR1) is not applicable. Assume there does not exist D' in $\mathsf{sub}(\mathcal{T})$ with $\mathcal{T} \models C \sqsubseteq \exists r.D'$ and $\mathcal{T} \models D' \sqsubseteq D$ but $\mathcal{T} \not\models D \sqsubseteq D'$. Then D is $\mathcal{T}$-equivalent to a $(\mathcal{T}, \Delta)$-indecomposable concept.

Proof of Claim. Take the finest signature Δ-decomposition $\Sigma_1, \ldots, \Sigma_n$ of $\mathcal{T}$ and assume $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are $\mathcal{EL}$-TBoxes that realize this decomposition. Then

$$\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n \models C \sqsubseteq \exists r.D$$

As (RR1) is not applicable by Lemma 44 there exists $\exists r.D' \in \mathsf{sub}(\mathcal{T}_i)$ for some $i \leq n$ such that

$$\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n \models C \sqsubseteq \exists r.D', \quad \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n \models D' \sqsubseteq D.$$

We show that D' is as required. D' (and even $\exists r.D'$) is $(\mathcal{T}, \Delta)$-indecomposable because $\exists r.D' \in \mathsf{sub}(\mathcal{T}_i)$. So, it is sufficient to show that $\mathcal{T} \models D \sqsubseteq D'$. We have $\mathcal{T} \models C \sqsubseteq \exists r.D'$. As (RR1) is not applicable and because Lemma 44, there exists $D'' \in \mathsf{sub}(\mathcal{T})$ such that $\models C \sqsubseteq \exists r.D''$ and $\mathcal{T} \models D'' \sqsubseteq D'$. Thus

$$\mathcal{T} \models D'' \sqsubseteq D$$

and we obtain, because of the conditions of the claim that $\mathcal{T} \models D'' \equiv D$. But then $\mathcal{T} \models D' \equiv D$. We have proved the claim. The proof shows that (R4) holds as well. ❑

Thus, in what follows we can work with TBoxes satisfying (R1)-(R4). Let $\mathcal{T}$ be a TBox satisfying (R1)-(R4). Then $\mathcal{T}$ satisfies (R5) if

- it contains no redundant axioms (i.e., if $\alpha \in \mathcal{T}$, then $\mathcal{T} \setminus \{\alpha\} \not\models \alpha$).
- if $C_1 \sqsubseteq C_2 \in \mathcal{T}$, $\mathsf{sig}(C_1) \not\subseteq \Delta$ and $\mathsf{sig}(C_2) \not\subseteq \Delta$, then there does not exist a concept D with $\mathsf{sig}(D) \subseteq \Delta$ such that $\mathcal{T} \models C_1 \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq C_2$.

We prove that we can work with TBoxes satisfying (R1)-(R5). First, we require the following

Lemma 49 *Assume C_0, C_1 are $\mathcal{T}$-equivalent to concepts C_0', C_1' that are $(\mathcal{T}, \Delta)$-indecomposable and that $\mathcal{T} \models C_0 \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq C_1$ where $\mathsf{sig}(D) \subseteq \Delta$. Let $\Delta' = \Delta \cup \{D_{C_0,C_1}\}$, where D_{C_0,C_1} is a fresh concept name, and let*

$$\mathcal{T}' = \{C_0 \sqsubseteq D_{C_0,C_1}, D_{C_0,C_1} \sqsubseteq C_1\}.$$

Then Σ_1, Σ_2 is a Δ-decomposition of $\mathcal{T}$ if, and only if, it is a Δ'-decomposition of $\mathcal{T}'$.

Proof. Assume $\mathcal{T}_1, \mathcal{T}_2$ realizes a Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$. Then $\mathsf{sig}(C_0') \subseteq \Sigma_i$ and $\mathsf{sig}(C_1') \subseteq \Sigma_j$ for some $i, j \in \{0,1\}$. We consider the case $\mathsf{sig}(C_0') \subseteq \Sigma_1$ and $\mathsf{sig}(C_1') \subseteq \Sigma_2$ and leave the remaining cases to the reader. Let

$$\mathcal{T}_1' = \mathcal{T}_1 \cup \{C_0' \sqsubseteq D_{C_0,C_1}\}, \quad \mathcal{T}_2' = \mathcal{T}_2 \cup \{D_{C_0,C_1} \sqsubseteq C_1'\}.$$

Then $\mathcal{T}' \equiv \mathcal{T}_1' \cup \mathcal{T}_2'$ and so $\mathcal{T}_1', \mathcal{T}_2'$ realizes the Δ'-decomposition Σ_1, Σ_2 of $\mathcal{T}'$, as required.

Conversely, let Σ_1, Σ_2 be a Δ'-decomposition of $\mathcal{T}'$. Take $\mathcal{T}_1, \mathcal{T}_2$ that realize this decomposition. Let

$$\mathcal{T}'' = \mathcal{T}' \cup \{D \equiv D_{C_0,C_1}\}$$

and

$$\mathcal{T}_1' = \mathcal{T}_1 \cup \{D \equiv D_{C_0,C_1}\}, \quad \mathcal{T}_2' = \mathcal{T}_2 \cup \{D \equiv D_{C_0,C_1}\}.$$

Then $\mathcal{T}_1', \mathcal{T}_2'$ realize the Δ'-decomposition Σ_1, Σ_2 of $\mathcal{T}''$. Now replace everywhere in $\mathcal{T}''$, $\mathcal{T}_1'$, and $\mathcal{T}_2'$, the concept D by D_{C_0,C_1} and denote the resulting TBoxes by $\mathcal{S}$, $\mathcal{S}_1$, and $\mathcal{S}_2$, respectively. Then

$$\mathcal{S} \equiv \mathcal{T} \equiv \mathcal{S}_1 \cup \mathcal{S}_2$$

and so $\mathcal{S}_1, \mathcal{S}_2$ realizes the Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}$, as required. ❏

Second, we require the following

Lemma 50 *Assume that $\mathcal{T}$ satisfies (R1)-(R4) and is role acyclic. If $C_1 \sqsubseteq C_2 \in \mathcal{T}$ is non-redundant, then C_1 is $\mathcal{T}$-equivalent to a $(\mathcal{T}, \Delta)$-indecomposable concept.*

Proof. Follows immediately from Lemma 53 below. ❏

We are now in the position to transform a TBox with (R1)-(R4) into a TBox with (R1)-(R5).

Theorem 51 *Suppose a TBox $\mathcal{T}$ satisfies (R1)-(R4). Then one can construct in polytime a TBox $\mathcal{T}'$ satisfying (R1)-(R5) and a $\Delta' \subseteq \Delta$ such that the Δ-decompositions of $\mathcal{T}$ coincide with the Δ'-decompositions of $\mathcal{T}'$.*

Proof. Clearly, by Lemma 49 and Lemma 50, it is sufficient to show that one can decide in polynomial time whether for given $\mathcal{T} \models C_1 \sqsubseteq C_2$ there exists a concept D with $\mathsf{sig}(D) \subseteq \Delta$ such that $\mathcal{T} \models C_1 \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq C_2$.

Assume $\mathcal{T} \models C_1 \sqsubseteq C_2$. Now, by Theorem 45, there exists such a D iff

$$\mathcal{T} \cup \mathcal{T}^{\Delta} \models C_1^{\Delta} \sqsubseteq C_2,$$

where $\mathcal{T}^{\Delta}$ and C_1^{Δ} are obtained from $\mathcal{T}$ and C_1 by replacing every non-Δ-symbol by a fresh symbols (simultaneously). Clearly, this condition is decidable in polynomial time. ❏

From now on we assume that the TBox $\mathcal{T}$ and Δ satisfy Conditions (R1)-(R5). Observe that we are now in a similar situation in $\mathcal{EL}$ as in our proof for DL-Lite when we had to move from DL-Lite$_{core}$ to DL-Lite$_{horn}$. The rewriting above took care of the right hand side of CIs, and now we have to consider complex left-hand sides. This is considerably harder for $\mathcal{EL}$ than for DL-Lite$_{horn}$, but the idea is the same. For a given $\alpha = C \sqsubseteq D \in \mathcal{T}$, we will be searching for a concept C' such that $\mathsf{sig}(C') \setminus \Delta \subseteq \mathsf{sig}(C) \setminus \Delta$ that is equivalent to C w.r.t. $\mathcal{T} \setminus \{\alpha\}$. We first prove a rather technical lemma that extends the proof of RJCP above. It will be convenient to have additional notation for signature renaming.

Assume $\mathcal{T}$ and Δ are fixed. We fix a signature Σ disjoint from Δ and consider fresh symbols X^o and X^* for $X \in \Sigma$. Set $\Sigma^o = \{X^o \mid X \in \Sigma\}$ and $\Sigma^* = \{X^* \mid X \in \Sigma\}$. For a concept C with $\mathsf{sig}(C) \subseteq \Sigma \cup \Sigma^o \cup \Sigma^* \cup \Delta$, we denote by C^o the resulting concept when every symbol $X \in \Sigma$ is replaced by X^o and every symbol X^* in C is replaced by X^o as well. C^* is defined in the same way, with $\cdot^o$ replaced by $\cdot^*$. We also set $r^o = r^* = r$ for all role names $r \in \Delta$. (Note that, by definition, $A^* = A^o = A$ for all $A \in \Delta$). Also, set

$$\mathcal{T}^o = \{C^o \sqsubseteq D^o \mid C \sqsubseteq D \in \mathcal{T}\},$$

$$\mathcal{T}^* = \{C^* \sqsubseteq D^* \mid C \sqsubseteq D \in \mathcal{T}\}.$$

For a signature Γ, TBox $\mathcal{T}'$, and concept C, we denote by

$$\mathsf{cons}_{\mathcal{T}',\Gamma}(C)$$

the set of $\mathcal{EL}$-concepts D such that $\mathcal{T}' \models C \sqsubseteq D$ and $\mathsf{sig}(D) \subseteq \Gamma$. The purpose of introducing $\mathsf{cons}_{\mathcal{T},\Gamma}(C)$ is similar to the purpose of $\mathsf{Cons}_{\mathcal{T}\setminus\{\alpha\},\Sigma\cup\Delta}(C)$ in our proof for DL-Lite$_{horn}$. Observe, however, that in contrast to DL-Lite without any restrictions this set is infinite. We use the notation "$\mathcal{T}', \Xi \models C$", with $\mathcal{T}'$ a TBox, Ξ a set of concepts, and C a concept, in the same way as introduced in the proof of Theorem 36.

Lemma 52 *Let $\mathcal{T}$ be a TBox, $\Delta \subseteq \mathsf{sig}(\mathcal{T})$, and $\mathsf{sig}(\mathcal{T}) \subseteq \Sigma \cup \Delta$. Let C be a concept with $\mathsf{sig}(C) \subseteq \Sigma^o \cup \Sigma^* \cup \Delta$.*

(i) The following holds for all D with $\mathsf{sig}(D) \subseteq \Sigma^o \cup \Delta$:

- *if $\mathcal{T}^o \cup \mathcal{T}^* \models C \sqsubseteq D$,*
- *then $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C))}(C^o) \models D$.*

(ii) The following holds for all D with $\mathsf{sig}(D) \subseteq \Sigma^* \cup \Delta$:

- *if $\mathcal{T}^o \cup \mathcal{T}^* \models C \sqsubseteq D$,*
- *then $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C))}(C^*) \models D$.*

Proof. (i) and (ii) are dual to each other; we have formulated both claims explicitly because the proof of (i) and (ii) is by *simultaneous* induction on the role depth of C. The case in which C is a conjunction of *concept names* is similar to the induction step and left to the reader.

For the induction step let C be the conjunction of

$$\bigsqcap_{i\in I_1} \exists r_i^o.C_i \sqcap \bigsqcap_{i\in I_2} A_i^o,$$

$$\bigsqcap_{i\in J_1} \exists r_i^*.C_i \sqcap \bigsqcap_{i\in J_2} A_i^*,$$

and

$$\bigsqcap_{i\in K_1} \exists r_i.C_i \sqcap \bigsqcap_{i\in K_2} A_i,$$

where

- r_i, $i \in K_1$ and A_i, $i \in K_2$, are Δ-symbols and
- r_i, $i \in I_1 \cup J_1$ and A_i, $i \in I_2 \cup J_2$, are non-Δ-symbols.

Note that C^o is the conjunction of

$$\bigsqcap_{i\in I_1} \exists r_i^o.C_i^o \sqcap \bigsqcap_{i\in I_2} A_i^o,$$

$$\bigsqcap_{i\in J_1} \exists r_i^o.C_i^o \sqcap \bigsqcap_{i\in J_2} A_i^o,$$

and

$$\bigsqcap_{i\in K_1} \exists r_i.C_i^o \sqcap \bigsqcap_{i\in K_2} A_i.$$

Note that the last conjunct equals

$$\bigsqcap_{i\in K_1} \exists r_i^o.C_i^o \sqcap \bigsqcap_{i\in K_2} A_i^o.$$

because $\cdot^o$ is the identity on Δ-symbols. As shown in (Lutz and Wolter 2010) and used in the proof of Theorem 36 already, we can take a tree-model $\mathcal{I}_o$ of $\mathcal{T}^o$ with root d_o such that the following conditions are equivalent for every $\mathcal{EL}$-concept D:

- $d_o \in D^{\mathcal{I}_o}$;
- $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C))}(C^o) \models D$.

Observe that there are d_i, $i \in I_1 \cup K_1$ with $(d_o, d_i) \in (r_i^o)^{\mathcal{I}_o}$ and $d_i \in (C_i^o)^{\mathcal{I}_o}$. Clearly, we may assume that the following conditions are equivalent for every $i \in I_1 \cup K_1$ and for every D:

- $d_i \in D^{\mathcal{I}_o}$
- $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C_i))}(C_i^o) \models D$

By induction hypothesis, the following holds for all $i \in I_1 \cup K_1$:

(i) for all D with $\mathsf{sig}(D) \subseteq \Sigma^o \cup \Delta$:

- if $\mathcal{T}^o \cup \mathcal{T}^* \models C_i \sqsubseteq D$,
- then $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C_i))}(C_i^o) \models D$,

and (ii) for all D with $\mathsf{sig}(D) \subseteq \Sigma^* \cup \Delta$:

- if $\mathcal{T}^o \cup \mathcal{T}^* \models C_i \sqsubseteq D$,
- then $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C_i))}(C_i^*) \models D$.

Thus, we find tree-like models $\mathcal{I}_i$ of $\mathcal{T}^o \cup \mathcal{T}^*$, $i \in I_1 \cup K_1$ with root c_i such that

- $c_i \in C_i^{\mathcal{I}_i}$;

and for all D with $\mathsf{sig}(D) \subseteq \Sigma^o \cup \Delta$

- if $c_i \in D^{\mathcal{I}_i}$, then $d_i \in D^{\mathcal{I}_o}$ (equivalently, $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C_i))}(C_i^o) \models D$)

and for all D with $\mathsf{sig}(D) \subseteq \Sigma^* \cup \Delta$

- if $c_i \in D^{\mathcal{I}_i}$, then $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C_i))}(C_i^*) \models D$.

For $d \in \Delta^{\mathcal{I}_o}$ denote by $\mathcal{I}_{\Sigma^o\cup\Delta}(d)$ the set of $\Sigma^o\cup\Delta$-concepts F with $d \in F^{\mathcal{I}}$. Now set

$$S = \{d \mid \exists r \in \Sigma^o \cup \Delta \;\; (d_o, d) \in r^{\mathcal{I}_o}\}$$

and take for every $d \in S$ a tree-like model $\mathcal{I}_d$ with root d such that

- $\mathcal{I}_d \models \mathcal{T}^o \cup \mathcal{T}^*$,
- for all concepts D the following conditions are equivalent:
  - $d \in D^{\mathcal{I}_d}$;
  - $\mathcal{T}^o \cup \mathcal{T}^*, \mathcal{I}_{\Sigma^o\cup\Delta}(d) \models D$.

Construct a new model $\mathcal{J}_o$ by setting

- $\Delta^{\mathcal{J}_o} = d_o \cup \bigcup_{i\in I_1\cup K_1} \Delta^{\mathcal{I}_i} \cup \bigcup_{d\in S} \Delta^{\mathcal{I}_d}$;
- for all $A \in \Delta \cup \Sigma^o \cup \Sigma^*$,
$$A^{\mathcal{J}_o} = \{d_0 \mid d_0 \in A^{\mathcal{I}_o}\} \cup \bigcup_{i\in I_1\cup K_1} A^{\mathcal{I}_i} \cup \bigcup_{d\in S} A^{\mathcal{I}_d};$$
- for all $r \in \Sigma^o \cup \Delta$, $r^{\mathcal{J}_o}$ is the union of
$$\{(d_o, c_i) \mid r = r_i^o, i \in I_1 \cup K_1\},$$
$$\{(d_o, d) \mid \exists r \in \Sigma^o \cup \Delta \;\; (d_o, d) \in r^{\mathcal{I}_o}\},$$
  and
$$\bigcup_{i\in I_1\cup K_1} r^{\mathcal{I}_i} \cup \bigcup_{d\in S} r^{\mathcal{I}_d}.$$
- for all $r \in \Sigma^*$,
$$r^{\mathcal{J}_o} = \bigcup_{i\in I_1\cup K_1} r^{\mathcal{I}_i} \cup \bigcup_{d\in S} r^{\mathcal{I}_d}.$$

The resulting interpretation has the following properties:

- the following conditions are equivalent for every D with $\mathsf{sig}(D) \subseteq \Sigma^o \cup \Delta$:
  - $d_o \in D^{\mathcal{J}_o}$
  - $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C))}(C^o) \models D$
- $\mathcal{J}_o$ is a model of $\mathcal{T}^o$;
- the restriction of $\mathcal{J}_o$ to $\Delta^{\mathcal{I}_0} \setminus \{d_o\}$ is a model of $\mathcal{T}^*$;
- d_o is an element of
$$(\bigsqcap_{i\in I_1} \exists r_i^o.C_i \sqcap \bigsqcap_{i\in I_2} A_i^o \sqcap \bigsqcap_{i\in K_1} \exists r_i.C_i \sqcap \bigsqcap_{i\in K_2} A_i)^{\mathcal{J}_o}.$$
- for every D with $\mathsf{sig}(D) \subseteq \Sigma^* \cup \Delta$:
  - if $d_o \in D^{\mathcal{J}_o}$
  - then $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C))}(C^*) \models D$

We prove the last statement and leave the proofs of the remaining statements to the reader. Let

$$D = \bigsqcap_{i\in S_1} \exists r_i^*.C_i^* \sqcap \bigsqcap_{i\in S_2} A_i^* \sqcap \bigsqcap_{i\in L_1} \exists r_i.C_i^*. \sqcap \bigsqcap_{i\in L_2} A_i$$

with $\mathsf{sig}(D) \subseteq \Sigma^* \cup \Delta$ and

- $r_i^* \in \Sigma^*$ for $i \in S_1$ and $A_i^* \in \Sigma^*$ for $i \in S_2$;
- $r_i \in \Delta$ for $i \in L_1$ and $A_i \in \Delta$ for $i \in L_2$.

Assume $d_o \in D^{\mathcal{J}_o}$. Then, clearly, $S_1 = \emptyset$ and $S_2 = \emptyset$. Now assume $i \in L_2$. Then $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C))}(C^o) \models A_i$ and so $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C))}(C^*) \models A_i$, as required.

Now consider an $\exists r_i.C_i^*$ with $i \in L_1$. Then $r_i \in \Delta$.

Case 1. $(d_o, d) \in r_i^{\mathcal{J}_o}$ and $d \in \Delta^{\mathcal{I}_o}$. Then

$$\mathcal{T}^* \cup \mathcal{T}^o, \mathcal{I}_{\Sigma^o\cup\Delta}(d) \models C_i^*.$$

By compactness, we can take an $F \in \mathcal{I}_{\Sigma^o\cup\Delta}(d)$ with $\mathcal{T}^* \cup \mathcal{T}^o \models F \sqsubseteq C_i^*$. By Theorem 45, we have a concept F_0 with $\mathsf{sig}(F_0) \subseteq \Delta$ such that $\mathcal{T}^o \models F \sqsubseteq F_0$ and $\mathcal{T}^* \models F_0 \sqsubseteq C_i^*$.

Thus, we have $d \in F_0^{\mathcal{I}_o}$. By definition of $\mathcal{I}_o$, $\exists r_i.F_0 \in \mathsf{cons}_{\mathcal{T}^o,\Delta}(C^o)$. Thus, $\exists r_i.F_0 \in \mathsf{cons}_{\mathcal{T}^*,\Delta}(C^*)$. In conclusion, $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta}(C^*) \models \exists r_i.C_i^*$, which implies, as required, $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C))}(C^*) \models \exists r_i.C_i^*$.

Case 2. $(d_o, c_j) \in r_i^{\mathcal{J}_o}$ for some $j \in K_1$ such that $c_j \in C_j^{\mathcal{I}_i}$ and $c_j \in (C_i^*)^{\mathcal{I}_i}$.

$$\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C_j))}(C_j^*) \models C_i^*.$$

But then

$$\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C))}(C^*) \models \exists r_i.C_i^*,$$

as required.

In the same way as $\mathcal{J}_o$, we construct a tree-like model $\mathcal{J}_*$ with root d_* and dual properties (obtained by swapping $\cdot^o$ and $\cdot^*$).

We may assume that $\mathcal{J}_*$ and $\mathcal{J}_o$ are disjoint except for $d_o = d_*$. Now take the union $\mathcal{J}$ of the two models. One can prove the following properties:

- $d_o \in C^{\mathcal{J}}$;
- $\mathcal{J}$ is a model of $\mathcal{T}^o \cup \mathcal{T}^*$;
- if $\mathsf{sig}(D) \subseteq \Sigma^o \cup \Delta$ and $d_o \in D^{\mathcal{J}}$, then $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C))}(C^o) \models D$.
- if $\mathsf{sig}(D) \subseteq \Sigma^* \cup \Delta$ and $d_o \in D^{\mathcal{J}}$, then $\mathcal{T}^*, \mathsf{cons}_{\mathcal{T}^*,\Delta\cup(\Sigma^*\cap\mathsf{sig}(C))}(C^*) \models D$.

Now assume that $\mathcal{T}^o \cup \mathcal{T}^* \models C \sqsubseteq D$, where $\mathsf{sig}(D) \subseteq \Sigma^o \cup \Delta$. Then $d_o \in D^{\mathcal{J}}$ and so, by Point 3, $\mathcal{T}^o, \mathsf{cons}_{\mathcal{T}^o,\Delta\cup(\Sigma^o\cap\mathsf{sig}(C))}(C^o) \models D$. This proves (i). The proof for (ii) is the same using Point 4 instead of Point 3. ❑

Lemma 53 *Assume $\mathcal{T}$ is role-acyclic. Let $\alpha = (C \sqsubseteq D) \in \mathcal{T}$ such that $\mathcal{T} \setminus \{\alpha\} \not\models \alpha$ and let Σ_1, Σ_2 be a Δ-decomposition of $\mathcal{T}$ such that*

- $\mathsf{sig}(D) \subseteq \Sigma_1$;
- $\mathsf{sig}(C) \cap \Sigma_1 \neq \emptyset$;
- $\mathsf{sig}(C) \cap \Sigma_2 \neq \emptyset$.

Then $\mathcal{T} \setminus \{\alpha\}, \mathsf{cons}_{\mathcal{T}\setminus\{\alpha\},(\Sigma_i\cap\mathsf{sig}(C))\cup\Delta}(C) \models C$ for some $i \in \{1, 2\}$.

Proof. Assume the lemma does not hold and let $\mathcal{T}$, Δ, $\alpha = C \sqsubseteq D$, and Σ_1, Σ_2 witness this. We will derive a contradiction. For a TBox $\mathcal{S}$ we denote by $\mathcal{S}_{\Sigma_1}$ the resulting TBox when all symbols X in Σ_2 are replaced by fresh symbols X^* in Σ_2^*. Similarly, we denote by $\mathcal{S}_{\Sigma_2}$ the resulting TBox when all symbols X in Σ_1 are replaced by fresh symbols X^* in Σ_1^*. The same notation is used for concepts and concept inclusions. We have

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \not\models C \sqsubseteq D$$

because $\mathcal{T} \setminus \{\alpha\} \not\models C \sqsubseteq D$. Since

$$\mathcal{T}_{\Sigma_1} \cup \mathcal{T}_{\Sigma_2} \models C \sqsubseteq D,$$

this implies

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \models C \sqsubseteq \exists r_1. \cdots \exists r_n.C_{\Sigma_i}$$

for some $i \in \{1, 2\}$ and some role names $r_1, \ldots, r_n$ and $n \geq 0$. But then, by the acyclicity condition above,

$$(\mathcal{T} \setminus \{\alpha\})_{\Sigma_1} \cup (\mathcal{T} \setminus \{\alpha\})_{\Sigma_2} \models C \sqsubseteq C_{\Sigma_i},$$

for some $i \in \{1, 2\}$. Using Lemma 52 and suitable renamings, this implies $\mathcal{T}\setminus\{\alpha\}, \mathsf{cons}_{\mathcal{T}\setminus\{\alpha\},(\Sigma_i\cap\mathsf{sig}(C))\cup\Delta}(C) \models C$ for some $i \in \{1, 2\}$. ❑

We now face the problem of working with $\mathsf{cons}_{\mathcal{T},\Sigma}(C)$. Observe that if $\mathcal{T}$ is role-acyclic, then $\mathsf{cons}_{\mathcal{T},\Sigma}(C)$ is equivalent to a *finite* set of concepts. Thus there exists a $\mathcal{EL}$-concept $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$ that is equivalent to $\mathsf{cons}_{\mathcal{T},\Sigma}(C)$; i.e., for every interpretation $\mathcal{I}$, we have $d \in (\mathsf{Cons}_{\mathcal{T},\Sigma}(C))^{\mathcal{I}}$ if, and only if, for all $F \in \mathsf{cons}_{\mathcal{T},\Sigma}(C)$ we have $d \in F^{\mathcal{I}}$. Unfortunately, $\mathsf{Cons}_{\mathcal{T},\Sigma}(C)$ can be of exponential size, and so we cannot employ it for a polytime algorithm without having a succinct representation. We now introduce a very convenient representation based on simulation quantifiers. The formalism is developed for role acyclic TBoxes but can be easily generalized to arbitrary TBoxes.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations and Σ a signature. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a Σ-*simulation* from $\mathcal{I}_1$ to $\mathcal{I}_2$ if the following holds:

- for all concept names $A \in \Sigma$ and all $(d_1, d_2) \in S$, if $d_1 \in A^{\mathcal{I}_1}$, then $d_2 \in A^{\mathcal{I}_2}$;
- for all role names $r \in \Sigma$, all $(d_1, d_2) \in S$, and all $e_1 \in \Delta^{\mathcal{I}_1}$ with $(d_1, e_1) \in r^{\mathcal{I}_1}$, there exists $e_2 \in \Delta^{\mathcal{I}_2}$ such that $(d_2, e_2) \in r^{\mathcal{I}_2}$ and $(e_1, e_2) \in S$.

If $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and there is a Σ-simulation S with $(d_1, d_2) \in S$, then $(\mathcal{I}_2, d_2)$ Σ-*simulates* $(\mathcal{I}_1, d_1)$, written $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$.

We now introduce the extension of $\mathcal{EL}$ with simulation quantifiers. We define $\mathcal{EL}^s$-concepts, concept inclusions, and TBoxes by simultaneous induction as follows. Concepts in $\mathcal{EL}^s$ are defined as follows:

- every $\mathcal{EL}$-concept (concept inclusion, TBox) is a $\mathcal{EL}^s$-concept (concept inclusion, TBox);
- if C is a $\mathcal{EL}^s$ concept, $\mathcal{T}$ is a $\mathcal{EL}^s$-TBox, and Σ a signature, then $D = \exists^{sim}\Sigma.(\mathcal{T}, C)$ is a $\mathcal{EL}^s$-concept;
- if C is a $\mathcal{EL}^s$ concept and D is a $\mathcal{EL}$-concept, then $C \sqsubseteq D$ is a $\mathcal{EL}^s$ concept inclusion;

- a finite set of $\mathcal{EL}^s$ concept inclusions is a $\mathcal{EL}^s$-TBox.

Let $\mathcal{I}$ be an interpretation. Then we let $d \in (\exists^{sim}\Sigma.(\mathcal{T}, C))^{\mathcal{I}}$ iff there exists an interpretation $\mathcal{J}$ that is a model of $\mathcal{T}$ with a $d' \in C^{\mathcal{J}}$ and $(\mathcal{J}, d') \leq_\Gamma (\mathcal{I}, d)$, where $\Gamma = (\mathsf{sig}(\mathcal{T}) \cup \mathsf{sig}(C)) \setminus \Sigma$.

The two main results we require are as follows:

Theorem 54 *Let $\mathcal{T}$ be role acyclic.*
(i) For all Σ and $\mathcal{EL}$-concepts C,

$$\exists^{sim}\Gamma.(\mathcal{T}, C) \equiv \mathsf{Cons}_{\mathcal{T},\Sigma}(C);$$

(i.e, $(\exists^{sim}\Gamma.(\mathcal{T}, C))^{\mathcal{I}} = (\mathsf{Cons}_{\mathcal{T},\Sigma}(C))^{\mathcal{I}}$ for all interpretations $\mathcal{I}$, where $\Gamma = (\mathsf{sig}(\mathcal{T}) \cup \mathsf{sig}(C)) \setminus \Sigma$.
(ii) Let $\mathcal{T}$ be an $\mathcal{EL}^s$-TBox and $C \sqsubseteq D$ a $\mathcal{EL}^s$-concept inclusion. Then the problem "$\mathcal{T} \models C \sqsubseteq D$" is decidable in polynomial time.

The first part is an immediate consequence of the well-known connection between simulations and $\mathcal{EL}$-concepts. The proof is standard, and therefore omitted, see e.g. (Lutz and Wolter 2010). Observe that as a consequence it is easily proved that every $\mathcal{EL}^s$-TBox (even with nested applications of simulation quantifiers) based on role-acyclic TBoxes is equivalent to the $\mathcal{EL}$-TBox obtained by replacing, recursively, simulation quantifiers $\exists^{sim}$ by the corresponding $\mathsf{Cons}(\cdot)$. Thus, the $\mathcal{EL}^s$-TBoxes we deal with in this paper inherit all properties we have established for $\mathcal{EL}$-TBoxes.

The proof of the second part is slightly more involved and uses a canonical model construction and the fact that it is decidable in poly-time whether $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$.

Finally, we are in the position to prove the second part of Theorem 27.

Theorem 27 (a) For role-acyclic $\mathcal{EL}$-TBoxes, the finest Δ-decomposition can be computed in polynomial time.

Proof. Let $\mathcal{T}$ be role-acyclic and $\Delta \subseteq \mathsf{sig}(\mathcal{T})$. We may assume that $\mathcal{T}$ satisfies conditions (R1)-(R5).

We now compute a $\mathcal{EL}^s$-TBox $\mathcal{T}'$ that is equivalent to $\mathcal{T}$ and such that $\mathsf{sdeco}_\Delta(\mathcal{T}')$ is the finest Δ-decomposition of $\mathcal{T}$. We proceed as follows: let $M = \{\alpha_1, \dots, \alpha_n\}$ be an enumeration of the $C \sqsubseteq D \in \mathcal{T}$ such that C contains at least two non-Δ-symbols.

1. For $1 \leq i \leq n$:
2. If $\mathcal{T} \setminus \{\alpha_i\} \models \alpha_i$, set $\mathcal{T} := \mathcal{T} \setminus \{\alpha_i\}$. Otherwise let $\alpha_i = C \sqsubseteq D$ and compute a minimal $\Sigma \subseteq \mathsf{sig}(C) \setminus \Delta$ such that
$$\mathcal{T} \setminus \{\alpha_i\} \models \exists^{sim}\Gamma_\Sigma.((\mathcal{T} \setminus \{\alpha_i\}), C) \sqsubseteq C$$
(we set $\Gamma_\Sigma = \mathsf{sig}(\mathcal{T}) \setminus (\Delta \cup \Sigma)$) and set $\mathcal{T} := \mathcal{T}$ if $\Sigma = \emptyset$, and
$$\mathcal{T} := (\mathcal{T} \setminus \{\alpha_i\}) \cup \{\exists^{sim}\Gamma_\Sigma.((\mathcal{T} \setminus \{\alpha_i\}), C) \sqsubseteq D\},$$
otherwise.
3. Let $\mathcal{T}' := \mathcal{T}$.

By definition, $\mathcal{T}'$ is logically equivalent to $\mathcal{T}$. Thus, it remains to observe that $\mathsf{sdeco}_\Delta(\mathcal{T}')$ is the finest Δ-decomposition of $\mathcal{T}'$. But this follows immediately from the first part of Theorem 54 and Lemma 53. Finally, $\mathcal{T}'$ can be computed in polytime by Theorem 54. ❏

Finally, we prove the first part of Theorem 27.

Theorem 27 (b) If $\Delta = \emptyset$, then the finest Δ-decomposition of any $\mathcal{EL}$-TBox can be computed in polynomial time.

Proof. Assume $\mathcal{T}$ is given and $\Delta = \emptyset$. We may assume that $\mathcal{T}$ satisfies conditions (R1)-(R4). Now we apply the following rule exhaustively to $\mathcal{T}$:

- If $C \sqsubseteq D \in \mathcal{T}$ and $\mathcal{T} \models C^\top \sqsubseteq D$, where $C^\top$ is obtained from C by replacing an occurrence of some subconcept ($\neq \top$) of C with $\top$, then $\mathcal{T} := \mathcal{T} \setminus \{\alpha\} \cup \{C^\top \sqsubseteq D\}$.

Call the resulting TBox $\mathcal{T}'$. Clearly $\mathcal{T}' \equiv \mathcal{T}$ and $\mathcal{T}'$ can be computed in polynomial time. We show that $\mathsf{sdeco}_\Delta(\mathcal{T}')$ is the finest Δ-decomposition of $\mathcal{T}$. Assume that this is not the case. Since $\mathcal{T}$ has properties (R1)-(R4), we then have a Δ-decomposition Σ_1, Σ_2 of $\mathcal{T}'$ and $C \sqsubseteq D \in \mathcal{T}'$ such that $\mathsf{sig}(D) \subseteq \Sigma_1$ but $\mathsf{sig}(C) \not\subseteq \Sigma_1$. We use the notation introduced in the proof of Lemma 53: so we have

$$\mathcal{T}'_{\Sigma_1} \cup \mathcal{T}'_{\Sigma_2} \models C \sqsubseteq D.$$

Since $\Delta = \emptyset$, it is readily checked that this implies $\mathcal{T}'_{\Sigma_1} \cup \mathcal{T}'_{\Sigma_2} \models C^* \sqsubseteq D$, where C^* is obtained from C by replacing any maximal subconcept $\exists r.C'$ with $r \in \Sigma_2$ by $\top$ and, in the resulting concept, any $A \in \Sigma_2$ by $\top$. Thus $\mathcal{T} \models C^* \sqsubseteq D$. But then the rewriting rule is applicable to $\mathcal{T}'$ and we have obtained a contradiction. ❏