The Algorithmic Complexity of Decomposability in Fragments of First-Order Logic*

Research Note

Denis Ponomaryov

Institute of Artificial Intelligence, University of Ulm, Germany;
A.P. Ershov Institute of Informatics Systems, Novosibirsk, Russia

ponom@iis.nsk.su

The research on applied logic in the mid 2000s has brought important methodological and algorithmic results on modularization of finitely axiomatizable logical theories. In particular, the research has been focused on the question whether logical theories have an inherent modular structure which could allow one to identify, in some sense, independent "components" in theories and use them as building blocks for creating new theories, or for improving efficiency of automated reasoning over theories [8, 3, 1, 6]. Remarkably, these research problems have revived practical interest to classical model-theoretic results related to interpolation and definability, conservative extensions and axiomatizable classes, which have found new applications in logics [8, 5, 9, 7]. One of the approaches based on results extending the well-known Craig interpolation was described in [12, 13] and was concerned with structuring logical theories into components sharing a given set of signature symbols. In these papers, the following notion of decomposability has been introduced (we use the notation $\mathtt{sig}(\mathcal{T})$ for the set of non-logical symbols of a theory $\mathcal{T}$):

Definition 1 *Let $\mathcal{T}$ be a theory and $\Delta \subseteq \mathtt{sig}(\mathcal{T})$ be a (possibly empty) subsignature. The theory $\mathcal{T}$ is called Δ–decomposable if there exist theories $\mathcal{S}_1$ and $\mathcal{S}_2$ such that:*

- $\mathtt{sig}(\mathcal{S}_1) \cup \mathtt{sig}(\mathcal{S}_2) = \mathtt{sig}(\mathcal{T})$;
- $\mathtt{sig}(\mathcal{S}_1) \cap \mathtt{sig}(\mathcal{S}_2) = \Delta$ *and* $\mathtt{sig}(\mathcal{S}_1) \neq \Delta \neq \mathtt{sig}(\mathcal{S}_2)$;
- $\mathcal{T}$ *is equivalent to* $\mathcal{S}_1 \cup \mathcal{S}_2$.

The theories $\mathcal{S}_1$ and $\mathcal{S}_2$ are called Δ–decomposition components of $\mathcal{T}$.

The algorithmic complexity of deciding the Δ–decomposability property has been studied in various calculi ranging from classical propositional [4] and description logics [7] to expressive fragments of first-order logic [11]. The results

* The author was supported by the German Research Foundation within the Transregional Collaborative Research Centre SFB/TRR 62 "Companion-Technology for Cognitive Technical Systems"

suggested that the complexity of decomposability coincides with the complexity of entailment in the underlying logic. Although this observation was not too surprising (since the definition of Δ–decomposability contains the logical equivalence), a general method for proving this claim was missing.

The aim of this paper is to describe a method for proving that the complexity of deciding decomposability coincides with the complexity of entailment in fragments of first-order logic. We illustrate this method by establishing the complexity of decomposability in the so-called signature fragments of first-order logic, i.e. those which are obtained by putting restrictions on signature (Theorem 1). The main constructions are given in Proposition 1 and the proof of point (2) of Theorem 1.

First, we introduce basic notations, auxiliary definitions and recall some well-known results from the computability theory.

We call a theory *finite* if it is finitely axiomatizable, i.e. defined by a single sentence. If $\mathcal{T}$ is a theory, then $\mathtt{sig}(\mathcal{T})$ denotes the *signature* of $\mathcal{T}$, i.e. the set of all non-logical symbols occurring in sentences of $\mathcal{T}$. The same notation $\mathtt{sig}(\varphi)$ will be used to denote the signature of a sentence φ. If Σ is a signature, then $|\Sigma|$ will denote the cardinality of Σ and for a structure $\mathfrak{M}$, we use $\mathtt{card}(\mathfrak{M})$ to denote the cardinality of the domain of $\mathfrak{M}$. If $\mathfrak{M}$ is a structure and Σ is a signature, then $\mathfrak{M}|_\Sigma$ will denote the restriction of $\mathfrak{M}$ onto Σ.

We omit subtleties related to effective representation of finite theories and signatures assuming there is a Gödel numbering of sentences and finite sets of non-logical symbols which assigns an index (a natural number) to every pair $\langle \mathcal{T}, \Delta \rangle$, where $\mathcal{T}$ is a finite theory and Δ is a finite signature.

Definition 2 *Let Σ and $\Delta \subseteq \Sigma$ be finite signatures. The Δ–decomposability problem for signature Σ is the set of indices of pairs $\langle \mathcal{T}, \Delta \rangle$, where $\mathcal{T}$ is a finite Δ–decomposable theory in signature Σ.*

In other words, this is the problem to decide whether a given finite set of sentences in signature Σ is Δ–decomposable. We will also consider the problem of deciding whether a finite theory $\mathcal{T}$ in a finite signature Σ given by a partition $\{\sigma_1, \sigma_2, \Delta\}$ is Δ–decomposable into some components in signatures $\sigma_1 \cup \Delta$ and $\sigma_2 \cup \Delta$, respectively. For brevity, we will refer to this as the problem to decide whether a given theory $\mathcal{T}$ is *Δ–decomposable with a partition* $\{\sigma_1, \sigma_2\}$.

Definition 3 *Two sets A and B are called effectively inseparable if there exists a 2-ary computable function h such that for any computably enumerable disjoint sets $W_x \supseteq A$ and $W_y \supseteq B$ it holds that $h(x, y) \in \omega \setminus (W_x \cup W_y)$.*

A theory $\mathcal{T}$ is called finitely inseparable if the set of valid sentences in signature $\mathtt{sig}(\mathcal{T})$ and the set of sentences whose negation holds in some finite model of $\mathcal{T}$ are effectively inseparable.

Lemma 1 ([10, Theorem 6.23]) *If A and B are computably enumerable and effectively inseparable then both A and B are Σ^0_1–complete.*

Definition 4 *We call a finite signature σ complex if either of the following holds:*

1. *σ contains at least one binary predicate;*
2. *σ contains at least one function of arity $\geqslant 2$;*
3. *σ contains at least two unary functions.*

Lemma 2 ([10, Theorems 16.52, 16.54, 16.58]) *A theory of complex signature is finitely inseparable.*

Since by definition, a complex signature σ may consist only of a single (predicate of function) symbol and any theory in such signature σ is not Δ–decomposable (for any subset $\Delta \subseteq \sigma$), point (1) of the main theorem below is formulated wrt extensions of complex signatures.

Theorem 1 *(1) For any complex signature σ, there exists a finite extension $\Sigma \supseteq \sigma$ such that the $\varnothing$–decomposability problem for Σ is undecidable.*

(2) For a finite signature Σ consisting of monadic predicates and constants it is coNEXPTIME-complete to decide whether a finite theory in signature Σ is Δ–decomposable with a given partition $\{\sigma_1, \sigma_2\}$.

Proof. Let us formulate several auxiliary lemmas and propositions.

Lemma 3 *Let $\mathcal{T}$ be a theory and $\{\sigma_1, \sigma_2, \Delta\}$ be a partition of the signature of $\mathcal{T}$. For $i = 1, 2$, denote by $\mathcal{T}_i$ the set of formulas in signature $\Sigma_i = \sigma_i \cup \Delta$ entailed by $\mathcal{T}$. Then the following conditions are equivalent:*

1. *$\mathcal{T}$ is Δ–decomposable into components in signatures Σ_1 and Σ_2, respectively;*
2. *if $\mathfrak{M}$ is a model in signature $\mathtt{sig}(\mathcal{T})$ and $\mathfrak{M}|_{\Sigma_i} \models \mathcal{T}_i$ for $i = 1, 2$, then $\mathfrak{M} \models \mathcal{T}$.*

Proof of the lemma. (1)$\Rightarrow$(2): If $\mathcal{T}$ is Δ–decomposable into some components in signatures Σ_1 and Σ_2, then $\mathcal{T}$ is Δ–decomposable into $\mathcal{T}_1$, $\mathcal{T}_2$, which trivially yields (2).

(2)$\Rightarrow$(1): we have $\mathcal{T} \models \mathcal{T}_1 \cup \mathcal{T}_2$, thus it suffices to show that $\mathcal{T}_1 \cup \mathcal{T}_2 \models \mathcal{T}$. Let $\mathfrak{M} \models \mathcal{T}_1 \cup \mathcal{T}_2$, then $\mathfrak{M}|_{\Sigma_i} \models \mathcal{T}_i$ for $i = 1, 2$ and hence, $\mathfrak{M} \models \mathcal{T}$. □

For a natural number n, denote by θ_n the equality sentence which holds only in models of cardinality $\geqslant n$.

Lemma 4 *Let φ be a sentence which is neither valid, nor unsatisfiable and assume that φ and $\neg\varphi$ do not have models of the same cardinality. Then either φ, or $\neg\varphi$ is equivalent to a sentence θ_n for some $n \in \omega$.*

Proof of the lemma. Consider the sets

$$D^+ = \{n \in \omega \mid \varphi \text{ has a model of cardinality } n\}$$
$$D^- = \{n \in \omega \mid \neg\varphi \text{ has a model of cardinality } n\}$$

It holds $D^+ \cup D^- = \omega$ and from the condition of the lemma we have $D^+ \cap D^- = \varnothing$. Clearly, one of the sets D^+ or D^- is finite, because otherwise φ and $\neg\varphi$ would have countable models. Assume that D^+ is finite, it is not hard to verify that the definition of this set yields that $\mathfrak{M} \models \varphi$ iff $\mathtt{card}\,(\mathfrak{M}) \in D^+$. Hence φ is equivalent to the disjunction $\bigvee_{n\in D^+} \theta_n$ and thus, is equivalent to the sentence θ_m, where m is the infimum of D^+. □

Proposition 1 *Let σ_1 and σ_2 be disjoint signatures such that $|\sigma_1| = 1$ and $1 \leqslant |\sigma_2| \leqslant 2$. Consider the theory $\mathcal{T} = \{\varphi \vee (\varphi' \wedge \psi') \vee (\neg\varphi' \wedge \psi'') \vee \psi\}$ such that $\varphi, \varphi', \psi', \psi'', \psi$ are sentences, $\mathtt{sig}\,(\mathcal{T}) = \sigma_1 \cup \sigma_2$, and the following conditions hold:*

- $\mathtt{sig}\,(\varphi) = \mathtt{sig}\,(\varphi') = \sigma_1$;
- $\mathtt{sig}\,(\psi) \subseteq \sigma_2$ *and* $\mathtt{sig}\,(\psi') = \mathtt{sig}\,(\psi'') \subseteq \sigma_2$;
- $|\mathtt{sig}\,(\psi)| = |\mathtt{sig}\,(\psi')| = |\mathtt{sig}\,(\psi'')| = 1$;
- *φ', ψ', ψ'', ψ and their negations have models of any cardinality;*
- *the sentence $\psi' \wedge \neg\psi''$ has models of any cardinality.*

Then the theory $\mathcal{T}$ is $\varnothing$–decomposable iff φ is valid.

Proof of the proposition. For the sake of clarity, we note that there exist signatures σ_1, σ_2 and sentences $\varphi', \psi', \psi'', \psi$ satisfying the conditions of the proposition. For instance, if σ_1 consists of a binary predicate R and $\sigma_2 = \{P, Q\}$, where P, Q are monadic predicates, then $\varphi' = \exists x, y R(x, y)$, $\psi' = \forall x P(x)$, $\psi'' = \exists x \neg P(x)$, $\psi = \exists x Q(x)$ are the sentences satisfying the above conditions. We also note that the presence of ψ in the definition of $\mathcal{T}$ is not important for proving the proposition (it holds for the theory $\mathcal{T}$ defined without the disjunct ψ), however it will be necessary for the proof of point (2) of Theorem 1.

If φ is valid then the theory $\mathcal{T}$ is tautological and hence, $\varnothing$–decomposable. Now assume that φ is not valid, but $\mathcal{T}$ is $\varnothing$–decomposable into some components $\mathcal{S}_1$ and $\mathcal{S}_2$ in signatures Σ_1 and Σ_2, respectively. We consider the three possible cases:

(A) $\Sigma_1 = \sigma_1$ and $\Sigma_2 = \sigma_2$;
(B) $\Sigma_1 = \sigma_1 \cup \mathtt{sig}\,(\psi')$ and $\Sigma_2 = \mathtt{sig}\,(\psi)$;
(C) $\Sigma_1 = \sigma_1 \cup \mathtt{sig}\,(\psi)$ and $\Sigma_2 = \mathtt{sig}\,(\psi')$;

Suppose that φ is satisfiable and assume first that φ and $\neg\varphi$ have models of the same cardinality. Suppose (A) holds and consider the sentences

$$\xi_1 = \neg\varphi \wedge (\varphi' \wedge \psi'), \quad \xi_2 = \varphi \wedge \neg\psi' \wedge \neg\psi$$

It follows from the conditions of proposition that the sentences $\varphi' \wedge \psi'$ and $\neg\psi' \wedge \neg\psi$ have models of any cardinality and ξ_1 is satisfiable iff so is $\neg\varphi \wedge \varphi'$. Assume this is the case. Then by our assumption on φ, there exist models $\mathfrak{M}_i \models \xi_i$, $i = 1, 2$ having a common domain. We have $\mathfrak{M}_1, \mathfrak{M}_2 \models \mathcal{T}$, hence, there exists a model $\mathfrak{M}$ such that $\mathfrak{M}\mid_{\Sigma_i} = \mathfrak{M}_i\mid_{\Sigma_i}$ for $i = 1, 2$. We have $\mathfrak{M}\mid_{\Sigma_i} \models \mathcal{S}_i$, thus by Lemma 3, we must have $\mathfrak{M} \models \mathcal{T}$, which is obviously not the case, since $\mathfrak{M} \models \neg\varphi \wedge \varphi' \wedge \neg\psi' \wedge \neg\psi$.

If $\neg\varphi \wedge \varphi'$ is unsatisfiable, then $\neg\varphi \wedge \neg\varphi'$ has a model and we consider the sentences

$$\xi_1 = \neg\varphi \wedge (\neg\varphi' \wedge \psi''), \quad \xi_2 = \varphi \wedge \neg\psi'' \wedge \neg\psi$$

By the assumption on φ and by the conditions of the proposition, we again conclude that there exist models $\mathfrak{M}_i \models \xi_i$, $i = 1, 2$ having a common domain, thus there is a model $\mathfrak{M}$ such that $\mathfrak{M} \mid_{\Sigma_i} = \mathfrak{M}_i \mid_{\Sigma_i}$ for $i = 1, 2$. We arrive at contradiction, since $\mathfrak{M} \models \neg\varphi \wedge \neg\varphi' \wedge \neg\psi'' \wedge \neg\psi$, and therefore conclude that case (A) is impossible.

No assume we have (B). Then similarly, we consider models of the sentences

$$\xi_1 = \neg\varphi \wedge ({}^{\varepsilon}\varphi' \wedge \neg\psi'^{\epsilon}) \wedge \psi, \quad \xi_2 = \varphi \wedge \neg\psi$$

having a common domain (where ε stands for a possible negation and ϵ stands for an additional $'$ in case $\neg\varphi \wedge \varphi'$ is unsatisfiable). Then again by using Lemma 3 we conclude that case (B) is impossible.

Finally, in case (C) it suffices to consider the models

$$\mathfrak{M}_1 = \neg\varphi \wedge ({}^{\varepsilon}\varphi' \wedge \psi'^{\epsilon}) \wedge \neg\psi, \quad \mathfrak{M}_2 = \varphi \wedge \neg\psi'^{\epsilon}$$

having a common domain to show that this case is impossible.

Now suppose that φ and $\neg\varphi$ do not have models of the same cardinality. Then by Lemma 4, the theory $\mathcal{T}$ is equivalent to $\theta \vee (\varphi' \wedge \psi') \vee (\neg\varphi \wedge \psi'') \vee \psi$, where θ is an equality sentence. It is proved analogously by considering cases (A)-(C) that $\mathcal{T}$ is not $\varnothing$–decomposable. For example, let us consider case (A). Take the formulas

$$\xi_1 = \neg\theta \wedge \neg\varphi' \wedge \psi, \quad \xi_2 = \neg\theta \wedge (\varphi' \wedge \psi') \wedge \neg\psi''$$

It follows from the conditions of the proposition that the sentences $\neg\varphi' \wedge \psi$ and $(\varphi' \wedge \psi') \wedge \neg\psi''$ have models of any cardinality. Therefore there exist models $\mathfrak{M}_i \models \xi_i$, $i = 1, 2$ having a common domain, which by Lemma 3 yields that case (A) is impossible.

If φ is unsatisfiable, then $\mathcal{T}$ is equivalent to $(\varphi' \wedge \psi') \vee (\neg\varphi \wedge \psi'') \vee \psi$ and it is proved analogously by considering cases (A)-(C) that $\mathcal{T}$ is not $\varnothing$–decomposable. □

Proposition 2 *Let $\mathcal{T}$ be a finite theory with $|\mathtt{sig}(\mathcal{T})| = 2$ and let $\mathcal{T}$ entail among equality formulas only tautologies. Assume $\mathcal{T}$ is $\varnothing$–decomposable into components $\mathcal{S}_1$ and $\mathcal{S}_2$ which are decidable theories.*

Then the following holds: if the $\varnothing$–decomposability problem is decidable for signature $\mathtt{sig}(\mathcal{T})$, then $\mathcal{T}$ is decidable.

Proof of the proposition. First let us note the following property ($*$): for any sentence φ, with $\mathtt{sig}(\varphi) \subseteq \mathtt{sig}(\mathcal{S}_i)$ for some $i = 1, 2$, we have $\mathcal{T} \models \varphi$ iff $\mathcal{S}_i \models \varphi$. For instance, let $\mathtt{sig}(\varphi) \subseteq \mathtt{sig}(\mathcal{S}_1)$. W.l.o.g. we may assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are finitely axiomatizable by some sentences ξ_1 and ξ_2, respectively. We have

$\xi_1, \xi_2 \models \varphi$, thus by interpolation there is an equality formula θ such that $\xi_2 \models \theta$ and $\theta \models \xi_1 \to \varphi$. By the conditions of the proposition, only tautological equality formulas are entailed by $\mathcal{T}$, so we obtain $\xi_1 \models \varphi$, i.e. $\mathcal{S}_1 \models \varphi$.

We now show how to obtain a decision procedure for $\mathcal{T}$. Let φ be an arbitrary sentence with $\mathtt{sig}(\varphi) \subseteq \mathtt{sig}(\mathcal{T})$. By property $(*)$ and the decidability of $\mathcal{S}_i$, $i = 1, 2$, it suffices to consider the case when $\mathtt{sig}(\varphi) = \mathtt{sig}(\mathcal{T})$. Consider the finite theory $\mathcal{T}' = \mathcal{T} \cup \{\varphi\}$. If $\mathcal{T}'$ is not $\varnothing$–decomposable then clearly, $\mathcal{T} \not\models \varphi$. If $\mathcal{T}'$ is $\varnothing$–decomposable, then we compute decomposition components as follows. Let ψ be a sentence which axiomatizes $\mathcal{T}'$. Computing decomposition components of $\mathcal{T}'$ reduces to finding sentences ψ_1 and ψ_2 such that $\mathtt{sig}(\psi_i) \subseteq \mathtt{sig}(\mathcal{S}_i)$, $\psi \models \psi_i$ and $\wedge\psi_i \models \psi$, for $i = 0, 1$. Since $\mathcal{T}'$ is known to be $\varnothing$–decomposable and the entailment relation is computably enumerable, this procedure is effective. Now it remains to note that $\mathcal{T} \models \varphi$ holds iff $\xi_i \models \psi_i$ for $i = 1, 2$. The "only if" direction is obvious and for the "if" direction note that $\mathcal{T} \models \varphi$ yields $\mathcal{T} = \mathcal{T}'$, hence $\mathcal{T}$ is equivalent to $\{\psi_0\} \cup \{\psi_1\}$ and property $(*)$ gives the required statement. $\square$

Let us complete the proof of Theorem 1.

(Point 1.) Consider the signature $\Sigma = \sigma_1 \cup \sigma_2$, where $\sigma_1 \cap \sigma_2 = \varnothing$, σ_1 consists of a predicate or a function symbol of arity $\geqslant 2$ and σ_2 is a non-empty signature. Then, by Lemmas 1, 2 and Proposition 1, the Σ^0_1–complete set of valid sentences in signature σ_1 is 1-reducible to the set of $\varnothing$-decomposable theories in signature Σ. Note also that the Δ-decomposability problem for a finite signature is computably enumerable which shows Σ^0_1–completeness.

Now assume that Σ consists of two unary function symbols and consider the set $\mathcal{T}$ of valid sentences in signature Σ. The theory $\mathcal{T}$ is finitely axiomatizable and $\varnothing$–decomposable into components $\mathcal{S}_i$, $i = 1, 2$ which are decidable theories (e.g. see [10], Chapter 13). By Propositions 2 and 1 we obtain the truth-table reducibility of the Σ^0_1–complete set of valid sentences in signature Σ to the set of $\varnothing$–decomposable theories in signature Σ.

(Point 2.) It is known that the set of valid formulas in a signature consisting of monadic predicates and constants (we call such signature *monadic*) is coNEXPTIME-complete [2]. In fact, we will present a construction for proving that the complexity of deciding Δ–decomposability with a given signature partition coincides with the complexity of the entailment relation in the underlying fragment of first-order logic. The construction uses the idea from Proposition 1. It also shows that deciding $\varnothing$–decomposability is at least as hard as deciding the entailment in the underlying fragment.

Let φ be a sentence in signature Σ and let $\Delta \subseteq \mathtt{sig}(\varphi)$ be a subsignature. We note the following first: the theory $\{\varphi\}$ is Δ–decomposable into some components in signatures Σ_1 and Σ_2, respectively, iff $\varphi|_{\Sigma_1} \wedge \varphi|_{\Sigma_2} \models \varphi$ holds, where $\mathtt{sig}(\varphi|_{\Sigma_1}) \cap \mathtt{sig}(\varphi|_{\Sigma_2}) = \Delta$ and for $i = 1, 2$, $\varphi|_{\Sigma_i}$ is a "copy" of φ, with all the symbols from $(\Sigma_{3-i} \setminus \Delta)$ injectively renamed into "fresh" ones, not present in $\mathtt{sig}(\varphi)$.

The "if" direction is obvious: if $\{\psi_1\}$ and $\{\psi_2\}$ are the corresponding decomposition components of $\{\varphi\}$, then $\varphi|_{\Sigma_i} \models \psi_i$, for $i = 1, 2$. The "only if" direction is a consequence of Lemma 1 in [13] (or Theorems 4 and 12 in [7]) and is proved by using interpolation as follows. We have $\varphi|_{\Sigma_1} \wedge \varphi|_{\Sigma_2} \models \varphi$, hence $\varphi|_{\Sigma_1} \models \varphi|_{\Sigma_2} \to \varphi$ and thus, there is a formula θ_1, with $\mathtt{sig}\,(\theta_1) \subseteq \Sigma_1$, such that $\varphi|_{\Sigma_1} \models \theta_1$ and $\theta_1 \wedge \varphi|_{\Sigma_2} \models \varphi$. Therefore, $\varphi|_{\Sigma_2} \models \theta_1 \to \varphi$ and there is a formula θ_2, with $\mathtt{sig}\,(\theta_2) \subseteq \Sigma_2$, such that $\varphi|_{\Sigma_2} \models \theta_2$ and $\theta_1 \wedge \theta_2 \models \varphi$. Since any model of φ expands to a model of $\varphi|_{\Sigma_i}$, for $i = 1, 2$, we have $\varphi \models \theta_i$ and hence, φ is equivalent to $\theta_1 \wedge \theta_2$. By the definition of θ_i, we conclude that $\{\varphi\}$ is Δ–decomposable into the components $\{\theta_1\}$ and $\{\theta_2\}$ in signatures Σ_1 and Σ_2, respectively.

It follows that $\{\varphi\}$ is Δ–decomposable iff there is a partition $\{\sigma, \sigma_2\}$ of signature $\mathtt{sig}\,(\varphi) \setminus \Delta$ such that for $\Sigma_i = \sigma_i \cup \Delta$, $i = 1, 2$, the formula $\varphi|_{\Sigma_1} \wedge \varphi|_{\Sigma_2} \to \varphi$ is valid, which shows that our problem is in coNEXPTIME.

Let us now prove coNEXPTIME-hardness. We will show that there exist a monadic signature Σ and a partition $\pi = \{\sigma_1, \sigma_2\}$ of Σ such that the set of valid sentences in a monadic signature containing at least one predicate 1-reduces to the set of finite theories in signature Σ, which are $\varnothing$-decomposable with the partition π. We will extend the construction from Proposition 1 (which works only for sentences in signatures of cardinality 1).

Let φ be a sentence in a monadic signature and let $\mathtt{sig}\,(\varphi)$ contain at least one monadic predicate. Consider the theory defined as $\mathcal{T} = \{\varphi \vee (\varphi' \wedge \psi') \vee (\neg\varphi' \wedge \psi'') \vee \psi\}$ and satisfying all the conditions of Proposition 1, except that φ', ψ', ψ'', and ψ now have monadic predicate signatures, $|\mathtt{sig}\,(\varphi)| \geqslant 1$, $\mathtt{sig}\,(\varphi') \subseteq \mathtt{sig}\,(\varphi)$, and $|\mathtt{sig}\,(\varphi')| = 1$. Define the theory

$$\mathcal{T}' = \mathcal{T} \cup \{ \bigwedge_{s \in \mathtt{sig}\,(\varphi)} (\mu(s) \vee \eta) \},$$

where for each $s \in \mathtt{sig}\,(\varphi)$, $\mu(s)$ is a sentence in signature $\{s\}$ and η is a sentence, with $|\mathtt{sig}\,(\eta)| = 1$, such that:

- $\mathtt{sig}\,(\eta) \cap \mathtt{sig}\,(\mathcal{T}) = \varnothing$;
- η and $\neg\eta$ have models of any cardinality;
- the sentence $\bigwedge_{s \in \mathtt{sig}\,(\varphi)} \mu(s)$ is satisfiable;
- every $\neg\mu(s)$, for $s \in \mathtt{sig}\,(\varphi)$, has models of any cardinality.

We claim that $\mathcal{T}'$ is $\varnothing$–decomposable with the signature partition $\{\mathtt{sig}\,(\psi') \cup \mathtt{sig}\,(\psi),\ \mathtt{sig}\,(\varphi) \cup \mathtt{sig}\,(\eta)\}$ iff φ is valid. The "only if" direction is trivial, since in this case $\mathcal{T}'$ is equivalent to the sentence $\chi \wedge \bigwedge_{s \in \mathtt{sig}\,(\varphi)} (\mu(s) \vee \eta)$, where χ is a tautological sentence in signature $\mathtt{sig}\,(\psi') \cup \mathtt{sig}\,(\psi)$. To prove the converse, assume $\neg\varphi$ is satisfiable and let us show that in this case $\mathcal{T}'$ is not $\varnothing$–decomposable. Note the following simple property which is a consequence of interpolation.

Lemma 5 *Let Δ be a signature and ψ_1, ψ_2 be sentences such that $\mathtt{sig}\,(\psi_1) \cap \mathtt{sig}\,(\psi_2) = \Delta$. Assume $\psi_1 \wedge \psi_2 \models \xi_1 \vee \xi_2$, where ξ_i is a sentence with $\mathtt{sig}\,(\xi_i) \subseteq \mathtt{sig}\,(\psi_i)$, for $i = 1, 2$. Then $\psi_1 \wedge \psi_2 \models (\xi_1 \vee \theta_\Delta)\ \wedge\ (\neg\theta_\Delta \vee \xi_2)$ for some sentence θ_Δ in signature Δ (in particular, if $\Delta = \varnothing$, then θ_Δ is an equality formula).*

Proof of the lemma. We have $\psi_1 \wedge \psi_2 \models \xi_1 \vee \xi_2$, hence $\psi_1 \wedge \neg\xi_1 \models \psi_2 \rightarrow \xi_2$ and by the conditions of the lemma, we have $\texttt{sig}\,(\psi_1 \wedge \neg\xi_1) \cap \texttt{sig}\,(\psi_2 \rightarrow \xi_2) = \Delta$. Then there exists a formula θ_Δ in signature Δ such that $\psi_1 \wedge \neg\xi_1 \models \theta_\Delta$ and $\theta_\Delta \models \psi_2 \rightarrow \xi_2$ hold. Therefore we obtain $\psi_1 \models \xi_1 \vee \theta_\Delta$, and $\psi_2 \models \neg\theta_\Delta \vee \xi_2$. $\square$

It follows from the lemma that if $\mathcal{T}'$ is $\varnothing$–decomposable, then all symbols from $\texttt{sig}\,(\varphi)$ must be contained with $\texttt{sig}\,(\eta)$ together in one decomposition component. For suppose the opposite, i.e. $\mathcal{T}'$ is $\varnothing$–decomposable into some components in signatures Σ_1 and Σ_2 and there is $s \in \texttt{sig}\,(\varphi)$ such that $s \in \Sigma_1$ and $\texttt{sig}\,(\eta) \subseteq \Sigma_2$. Then by the lemma above we have $\mathcal{T}' \models (\mu(s) \vee \theta) \wedge (\neg\theta \vee \eta)$, where θ is an equality formula. Assume $\neg\theta$ is satisfiable. By our conditions, the sentences ψ, η, and $\neg\mu(s)$ have disjoint signatures and have models of any cardinality. Therefore, there exists a structure $\mathfrak{M} \models \psi \wedge \eta \wedge \neg\mu(s) \wedge \neg\theta$ which is a model of $\mathcal{T}'$, but this contradicts the entailment $\mathcal{T}' \models \mu(s) \vee \theta$. If $\neg\theta$ is not satisfiable, then we have $\mathcal{T}' \models \eta$, which is again not the case, because there is a model $\mathfrak{M}$ of $\mathcal{T}'$ such that $\mathfrak{M} \models \psi \wedge \neg\eta \wedge \bigwedge_{s \in \texttt{sig}\,(\varphi)} \mu(s)$.

Since all the signature symbols of $\texttt{sig}\,(\varphi)$ must then belong to one decomposition component together with the symbol from $\texttt{sig}\,(\eta)$, the proof of Proposition 1 shows that $\mathcal{T}'$ is not $\varnothing$–decomposable, because w.l.o.g. we may now assume that $|\texttt{sig}\,(\varphi)| = 1$ and every model of $\mathcal{T}$ considered in the proof of Proposition 1 expands to a model of $\mathcal{T}'$ by interpreting the symbol from $\texttt{sig}\,(\eta)$ such that η is satisfied. This proves coNEXPTIME-hardness and concludes the proof of Theorem 1. $\square$

References

1. Amir E. and McIlraith S. Partition-based logical reasoning for first-order and propositional theories. Artificial Intelligence, 162, No. 1-2, pp. 49-88, 2005.
2. Bachmair L., Ganzinger H., and Waldmann U. Set Constraints are the Monadic Class. In: Proc. Eighth Annual IEEE Symposium on Logic in Computer Science, pp. 75-83, Montreal, Canada, June 19-23, 1993. IEEE Computer Society Press.
3. Cuenca Grau B., Horrocks I., Kazakov Y., and Sattler U. Modular Reuse of Ontologies: Theory and Practice. JAIR, 31, pp. 273-318.
4. Emelyanov P. and Ponomaryov D. The Complexity of AND-decomposition of Boolean Formulas. Manuscript. Available at http://persons.iis.nsk.su/en/person/ponom/papers
5. Ghilardi S., Lutz C., Wolter F., Zakharyaschev M. Conservative Extensions in Modal Logic. In: Proc. AiML, 2006.
6. Konev B., Lutz C., Walther D., and Wolter F. Semantic Modularity and Module Extraction in Description Logic. In: Proc. ECAI, 2008.
7. Konev B., Lutz C., Ponomaryov D., and Wolter F.: Decomposing Description Logic Ontologies. In: Proc. Twelfth International Conference on the Principles of Knowledge Representation and Reasoning (KR 2010).
8. Konev B., Lutz C., Walther D., and Wolter F.: Formal Properties of Modularisation. In: Ontology Modularisation. Parent C., Spaccapietra S., Stuckenschmidt H. (eds.), Springer, 2009.

9. Kourousias G. and Makinson D. Parallel Interpolation, Splitting, and Relevance in Belief Change. J. of Symb. Logic, 72, pp. 994-1002, 2007.
10. Monk J.D., Mathematical Logic. – Springer–Verlag, 1976.
11. Morozov A. and Ponomaryov D. On Decidability of the Decomposability Problem for Finite Theories. Sib. Math. J., 51, No. 4, pp. 667-674, 2010.
12. Ponomaryov D. A Decomposability Criterion for Elementary Theories. Sib. Math. J., 49, No. 1, pp. 152-154, 2008.
13. Ponomaryov D. On Decomposability in Logical Calculi. Bulletin of the Novosibirsk Computing Center, 28, pp. 111-120, 2008.