A Decomposability Criterion for Elementary Theories *

Denis Ponomaryov

Abstract

We prove that each elementary theory has a unique decomposition into indecomposable components and formulate a decomposability criterion.

Definition 1 *A theory $\mathcal{T}$ of signature Σ is called* ***decomposable****, if $\mathcal{T}$ is the deductive closure in the predicate calculus of signature Σ of all sentences of some theories $\mathcal{S}_1$ and $\mathcal{S}_2$ with the disjoint signatures Σ_1 and Σ_2, $\Sigma_1 \cup \Sigma_2 = \Sigma$ (we use the notation: $\mathcal{T} = \mathcal{S}_1 \uplus \mathcal{S}_2$).*

The theories $\mathcal{S}_1$ and $\mathcal{S}_2$ are called (decomposition) ***components*** *of $\mathcal{T}$.*

Only nontrivial decompositions, with $\Sigma_1 \neq \varnothing \neq \Sigma_2$, are of interest for consideration. Throughout this paper, we assume that every decomposition component of a theory $\mathcal{T}$ includes all equality formulas of $\mathcal{T}$. Thus every component $\mathcal{S}_i$ of signature Σ_i contains all sentences of $\mathcal{T}$ in signature Σ_i. For instance, if Σ consists of a sole symbol then every theory in this signature has only trivial decomposition.

Let us formulate the main question under study: *Consider a theory $\mathcal{T}$ of signature Σ defined by some set of axioms Φ in signature Σ. How can we determine whether $\mathcal{T}$ is decomposable judging from Φ?*

This question was formulated by D. Palchunov in [4]. The interest in this problem is connected with applications in computer science such as automated theorem proving [1] and the maintenance of terminological systems [3, 5].

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We answer the question as follows: First, we introduce the notion of decomposable sentence in Section 1 and demonstrate that the sentences of this kind are crucial for determining the decomposition components of a theory. Next, we prove that each theory has a unique decomposition into indecomposable components. The key result used in the proof is the well-known Craig interpolation theorem [2].

In Section 2 we describe a method of finding decomposition components for a given theory. This method makes it possible to formulate a decomposability criterion.

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1 The Theorem of the Uniqueness of the Decomposition

Proposition 1 *Let $\mathcal{P}$ and $\mathcal{Q}$ be theories of disjoint signatures $\Sigma_1 \cup \Sigma_2 = \Sigma$, and let φ be a sentence of signature Σ.*

If φ follows from the union of $\mathcal{P}$ and $\mathcal{Q}$, then there exist sentences $\theta \in \mathcal{P}$ and $\phi \in \mathcal{Q}$ such that $\theta, \phi \vdash \varphi$. Moreover, θ includes only those symbols of Σ_1 that are contained in φ, and ϕ includes only those symbols of Σ_2 that are present in φ.

As $\mathcal{P}, \mathcal{Q} \vdash \varphi$, there exist sentences $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, for which $P, Q \vdash \varphi$. Hence, $P \vdash Q \to \varphi$.

By Craig's interpolation theorem, there exists a sentence θ of signature Σ_1, which includes only those symbols of Σ_1 that are present in φ. Moreover, $P \vdash \theta$ and $\theta \vdash Q \to \varphi$. Hence, $Q \vdash \theta \to \varphi$.

Similarly, there exists a sentence ϕ of signature Σ_2, which includes only those symbols of Σ_2 that are contained in φ. Moreover, $Q \vdash \phi$ and $\phi \vdash \theta \to \varphi$. Thus, $\theta, \phi \vdash \varphi$. $\square$

Definition 2 *Let $\mathcal{T}$ be a theory. We call a sentence $\varphi \in \mathcal{T}$* ***decomposable in*** *$\mathcal{T}$ if there exist sentences $\theta \in \mathcal{T}$ and $\psi \in \mathcal{T}$ with the following properties: θ and ψ contain symbols only from the signature of φ; θ and ψ do not have signature symbols in common; neither θ nor ψ is an equality formula; $\theta, \psi \vdash \varphi$.*

Sentences θ and ψ are called ***decomposition fragments*** *of φ. If there are no such sentences in $\mathcal{T}$ then we call φ* ***indecomposable in*** $\mathcal{T}$.

Lemma 1 *Let $\mathcal{T}$ be a theory. For each sentence $\varphi \in \mathcal{T}$ there exist a sequence $\phi_1, \ldots, \phi_n$ of sentences such that $\phi_1, \ldots, \phi_n \vdash \varphi$ holds, and each ϕ_i, $i = 1 \ldots n$, is a sentence of $\mathcal{T}$ indecomposable in $\mathcal{T}$.*

Consider the set $T_1 = \{\varphi\}$. Take the decomposition fragments ϕ and ψ for φ, if they exist in $\mathcal{T}$, and build the set $T_2 = \{\phi, \psi\}$. By repeating this transformation for the sentences of T_2 and further resulting sets, we obtain the sequence $T_1, T_2, T_3, \ldots$. Each sentence contains only finitely many signature symbols; therefore, each sentence can be decomposed only finitely many times. Thus, for some k the set $T_k = \{\phi_1, \ldots, \phi_n\}$ will contain only those sentences of $\mathcal{T}$ that are indecomposable in $\mathcal{T}$, and for which $\phi_1, \ldots, \phi_n \vdash \varphi$ holds. $\square$

Theorem 1 *Let $\mathcal{T}$ be a theory of signature Σ. Then $\mathcal{T}$ has a unique decomposition into indecomposable components.*

More precisely, there exists a unique partition Π of Σ such that $\mathcal{T} = \biguplus\{\mathcal{T}_\sigma \mid \sigma \in \Pi\}$, with every $\mathcal{T}_\sigma$ a theory, which consists of all sentences of $\mathcal{T}$ in signature σ and has only trivial decompositions.

Let $\mathcal{S}_1 \subseteq \mathcal{T}$ be a theory of signature $\Sigma_1 \subseteq \Sigma$ consisting of all sentences of $\mathcal{T}$ in Σ_1. Then $\mathcal{S}_1$ is a decomposition component of $\mathcal{T}$ iff the following condition $(*)$ is satisfied: if φ is a sentence of signature $\Sigma_\varphi \cap \Sigma_1 \neq \varnothing$ and φ is indecomposable in $\mathcal{T}$ then $\Sigma_\varphi \subseteq \Sigma_1$.

We now prove this statement.

$\Rightarrow$: Let $\mathcal{T} = \mathcal{S}_1 \uplus \mathcal{S}_2$, where $\mathcal{S}_2$ is a theory of signature $\Sigma_2 = \Sigma \backslash \Sigma_1$. Suppose that $\Sigma_\varphi \cap \Sigma_2 \neq \varnothing$. Then $\mathcal{S}_1, \mathcal{S}_2 \vdash \varphi$. It follows from Proposition 1 that φ is decomposable in $\mathcal{T}$, which contradicts the initial assumption in $(*)$.

$\Leftarrow$: Let $\mathcal{S}_2$ be a theory of signature $\Sigma_2 = \Sigma \backslash \Sigma_1$, which consists of all sentences of $\mathcal{T}$ in signature Σ_2. Let ψ be a sentence of $\mathcal{T}$. By Lemma 1, there exists a sequence $\phi_1, \ldots, \phi_n$ of sentences such that $\phi_1, \ldots, \phi_n \vdash \psi$, and each ϕ_i, $i = 1 \ldots n$, is a sentence of $\mathcal{T}$ indecomposable in $\mathcal{T}$.

From $(*)$ we have $\{\phi_1, \ldots, \phi_n\} \subset \mathcal{S}_1 \cup \mathcal{S}_2$; hence, $\mathcal{S}_1 \uplus \mathcal{S}_2 \vdash \psi$.

Thus, a subset $\Sigma_1 \subseteq \Sigma$ corresponds to an indecomposable component of $\mathcal{T}$ iff it satisfies the condition $(*)$ and does not have a proper subset satisfying $(*)$. Note that the collection of subsets of Σ with the property $(*)$ is closed under intersection; thus, each symbol of Σ is contained in one minimal subset of Σ satisfying $(*)$, and these minimal subsets do not intersect. This completes the proof of Theorem 1. $\square$

2 A Decomposability Criterion

It follows from the proof of Theorem 1 that each sentence of a theory $\mathcal{T}$, which is indecomposable in $\mathcal{T}$, contains symbols only from one decomposition component of $\mathcal{T}$. This allows us to determine the partitioning of the signature, as well as components of $\mathcal{T}$ judging from the system of axioms of $\mathcal{T}$.

Let us formalize this result with the help of the following

Definition 3 *Let $\mathcal{T}$ be a theory of signature Σ, and let Φ be a system of axioms of $\mathcal{T}$.*

We call a pair of symbols $p, q \in \Sigma$ ***directly connected*** *(by Φ), if there exists a sentence $\psi \in \Phi$ containing p and q.*

The symbols p and q are called ***connected****, if there exists a sequence $p = t_1, \ldots, t_k = q$ of signature symbols in which every pair t_i, t_{i+1} is directly connected.*

Thus, for a given system Φ of axioms we have a nonoriented labelled graph, where the set of vertices is Σ, and the incidence relation is determined by sentences from Φ. The connectedness relation is an equivalence on Σ; therefore, the signature Σ is partitioned into the cosets, which coincide with the connectedness components of the graph. We may say that Φ *induces connectedness components on* Σ.

Remark 1 *Each theory $\mathcal{T}$ has a system of axioms consisting of sentences indecomposable in $\mathcal{T}$.*

Remark 2 *Let $\mathcal{T}$ be a theory of signature Σ and let Φ be a system of axioms for $\mathcal{T}$, with each sentence $\varphi \in \Phi$ indecomposable in $\mathcal{T}$.*

Then Φ induces a connectedness component $\sigma \subseteq \Sigma$ on Σ iff $\sigma \in \Pi$, where Π is a partition of Σ, which corresponds to the decomposition of $\mathcal{T}$ into indecomposable components.

We obtain from Theorem 1 that each system Φ of axioms of a theory $\mathcal{T}$, with all $\varphi \in \Phi$ indecomposable in $\mathcal{T}$, induces the same connectedness components on the signature of $\mathcal{T}$. This leads us to the following

Decomposability Criterion *A theory $\mathcal{T}$ of signature Σ is decomposable iff some system Φ of axioms of $\mathcal{T}$, with all $\varphi \in \Phi$ indecomposable in $\mathcal{T}$, induces more than one connectedness component on Σ.*

References

[1] E. Amir and S. McIlraith. Partition-based logical reasoning for first-order and propositional theories. *Artificial Intelligence*, 162(1-2):49-88, 2005.

[2] C.C. Chang and H.J. Keisler. *Model Theory.* North-Holland, 1973.

[3] B. Cuenca-Grau, B. Parsia, E. Sirin, and A. Kalyanpur. Modularity and Web ontologies. In *Proceedings of KR2006*, pages 198-208, 2006.

[4] D. Palchunov. GABEK for ontology generation. *GABEK. Contributions to Knowledge Organization*, 2, Wien: LIT-publishing Company, 2005.

[5] C. Lutz, D. Walther, and F. Wolter. Conservative extensions in expressive description logics. In *Proceedings of IJCAI 2007*, pages 453-458, 2007.