A Relational Solver for Constraint-based Type Inference

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$\lambda^a\mathcal{M}^a$ Programming Language

$\lambda^a\mathcal{M}^a$ — educational programming language:

- Functional: every program is expression, has first-class functions
- Imperative: expressions evaluate in strictly defined order, function calls can produce side effects, data structures mutation allowed
- Native-compiled: programs must be compiled before running, target platform is Intel x86
- Modular: programs can be composed from several modules, that are compiled separately
- Untyped: there aren't statical and runtime type checks

$\lambda^a\mathcal{M}^a$: Simple Program

This program calculates the Ackermann function of $0 \leq m \leq 3$ and $0 \leq n \leq 8$ by definition:

Here we may notice:

- variable definition, using and assignment
- function definition and calls (including recursive)
- conditional and loop expressions
- infix operators (including ";" standing for sequential evaluation operator)

```
var x, m, n ;

fun ack (m, n) {
  if m == 0 then n + 1
  elif m > 0 && n == 0 then ack (m - 1, 1)
  else ack (m - 1, ack (m, n - 1))
  fi
}

x := read () ;
for m := 0, m <= 3, m := m + 1 do
  for n := 0, n <= 8, n := n + 1 do
    write (ack (m, n))
  od
od
```

$\lambda^a\mathcal{M}^a$: Complex Data Structures

The following program generates list of 1000 integers and sorts it using bubble sort:

Here we may notice:

- arrays and linked lists
- pattern matching
- subvalues accessing by index

Linked list is the special case of more general construction, that in $\lambda^a\mathcal{M}^a$ named S-expression

```
fun compare (x, y) { x - y }
fun bubbleSort (l) {
  fun inner (l) {
    case l of
      x : z@(y : tl) ->
       if compare (x, y) > 0
       then [true, y : inner (x : tl) [1]]
       else case inner (z) of [f, z] -> [f, x : z] esac
       fi
    | _ -> [false, l]
    esac
  }
  fun rec (l) {
    case inner (l) of
      [true , l] -> rec (l)
    | [false, l] -> l
    esac
  }
  rec (l)
}
fun generate (n) { if n then n : generate (n - 1) else {} fi }
bubbleSort (generate (1000))
```

$\lambda^a\mathcal{M}^a$: S-expressions and First-Class Functions

The next example is simple λ-calculus interpreter:

We may notice:

- module importing
- first-class functions
- strings
- S-expressions

S-expressions acts like tagged arrays and provides convenient ADT-style data manipulation (like in HASKELL, OCAML, etc)

```
import Data ;

fun emptyContext () { fun (v) {
  failure ("No variable %s in current scope!\n", v.string)
} }
fun extendContext (ctx, v, x) {
  fun (u) { if v === u then x else ctx (u) fi }
}

fun eval (ctx, expr) {
  case expr of
    Val (x) -> x
  | Var (v) -> ctx (v)
  | App (f, x) -> eval (ctx, f) (eval (ctx, x))
  | Lam (v, b) ->
      fun (x) { eval (extendContext (ctx, v, x), b) }
  esac
}
fun eval0 (expr) { eval (emptyContext (), expr) }

var test = App (Lam ("x", Var ("x")), Val ("Hello!")) ;
printf ("%s\n", eval0 (test))
```

$\lambda^a\mathcal{M}^a$: Functional Programming, State Monad

- Since we have in $\lambda^a\mathcal{M}^a$ first-class functions support, sometimes it's convenient to use FP idioms like monads
- For example, to implement backtracking in parser it is convenient to use the state monad
- So, there is an implementation of this monad in the $\lambda^a\mathcal{M}^a$ standard library[1]:
- Also, we may notice utilization of the user-defined infix operators feature of $\lambda^a\mathcal{M}^a$

```
import Fun ;

infix >>= before $ (m, k) {
  fun (state) {
    case m (state) of
      [state, x] -> k (x) (state)
    esac
  }
}

fun pure (x) {
  fun (state) { [state, x] }
}
```

[1] The code is slightly changed for a reader convenience

Defects in $\lambda^a\mathcal{M}^a$ Programs

```
fun size (xs) {
  case xs of
    {} -> 0
  | _ : xs -> 1 + size (xs)
  esac
}

-- ...

size ("123")
```

(a) Simple defect

```
fun nextStep (x) { x + 1 }

-- ...

var myComputation = pure (1) >>= nextStep ;

-- ...

var initialState = 0 ;
myComputation (initialState) [1]
```

(b) Difficult defect

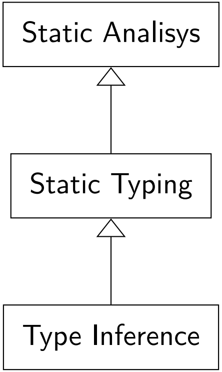
a) A string is passed instead of a list in "`size`":
  - 👍 Match failures are detected as soon as possible with detailed error messages
  - 👍 The cause of a crash is located immediately next to the crash location

b) A programmer forgot to wrap a result of "`nextStep`" in "`pure`":
  - 👎 Calling non-callable values crashes with the "Segmentation fault" error
  - 👎 The cause of a crash may be "across the code" from the crash location

Static Analysis and Static Typing

- **Static analysis** is a method to prevent some classes of program defects before running the program (ahead-of-time)
- A classical way to deal with defects like (b) is **static typing**
- In various modern programming languages, optional explicit type annotations were invented to allow static analysis through static typing (PYTHON, TYPESCRIPT, etc.)
  - 👍 Provides additional documentation
  - 👍 Simplifies static analyzers
  - 👎 Requires to change an existing code to use static analyzer on
  - 👎 Complicates language syntax
  - 👎 Distracts programmer

👍 Instead, we study static typing through full **type inference**, that allows to analyze programs without changing them



Constraint-based Type Inference

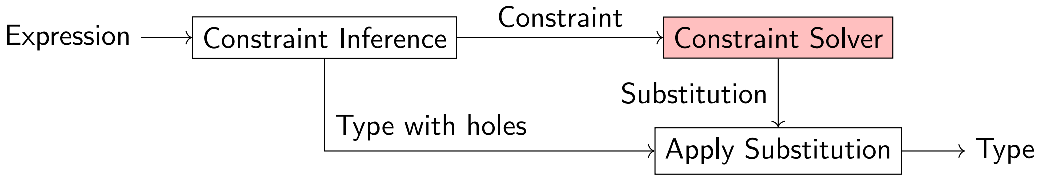
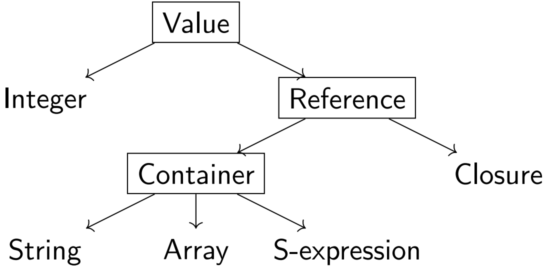


Figure: Constraint-based type inference

- Modern programming languages that support full type inference uses **constraint-based type inference**
- Currently, there are many frameworks that provides unified way to implement constraint-based type inference algorithm for given type system and constraint solver
- Thus, we need to develop the type system and the constraint solver

$\lambda^a\mathcal{M}^a$: Data Shapes Classification



(a) Shapes of data in $\lambda^a\mathcal{M}^a$

The shapes of data:

- Integer — integral value in bounds $[-2^{30}, 2^{30} - 1]$
- String — array of ASCII characters
- Array — array of arbitrary values
- S-expression — array with associated literal tag
- Closure — first-order function

Type System

$$\mathsf{T} ::= \mathbb{Z} \mid \mathbb{S} \mid [\mathsf{T}] \mid \mathcal{X}\ (\mathsf{T}, ..., \mathsf{T}) \sqcup ... \sqcup \mathcal{X}\ (\mathsf{T}, ..., \mathsf{T})$$
$$\mid \mathcal{X} \mid \forall \mathcal{X}, ..., \mathcal{X}.\ C \Rightarrow (\mathsf{T}, ..., \mathsf{T}) \to \mathsf{T} \mid \mu \mathcal{X}.\ \mathsf{T}$$

Legend:

- $\mathcal{X}$ — type variables and literal tags
- T — types
- C — constraints

Figure: Syntax of types

The forms of types:

- Integer, String
- Array, S-expression
- Type variable, Function
- Recursive type

- S-expression types are composition of product and sum types
- Function types are generic (by "$\mathcal{X}, ..., \mathcal{X}$") and possible specializations are constrained (by "C")
- We need to allow explicit recursive types to support recursive data structures, since there aren't any type declarations in source code

Type System: Examples

Obvious:

- `0` : $\mathbb{Z}$
- `123` : $\mathbb{Z}$
- `"123"` : $\mathbb{S}$
- `[]` : [T] for any `T`
- `[1, 2, 3]` : $[\mathbb{Z}]$
- `One (Two, Three)` : $One(Two, Three)$
- `One (Two, Three)` : $One(Two \sqcup A, Three \sqcup B \sqcup C) \sqcup D$, since S-expression types may include more that one constructor

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But:

- `0` : T for any T, since "`0`" is used as null value in many cases
- `{}` : T for any T, since "`{}`" is a syntactic sugar for "`0`"
- `{1, 2, 3}` : $cons(\mathbb{Z}, cons(\mathbb{Z}, cons(\mathbb{Z}, \mathsf{T})))$ for any T, as in previous case
- `{1, 2, 3}` : $\mu\alpha.\ cons(\mathbb{Z}, \alpha)$, since this is a general type of all lists of integers

Constraint System

$$C ::= \top \mid C \wedge C \mid Ind(\mathsf{T}, \mathsf{T}) \mid Call(\mathsf{T}, \mathsf{T}, ..., \mathsf{T}, \mathsf{T}) \mid Sexp_{\mathcal{X}}(\mathsf{T}, \mathsf{T}, ..., \mathsf{T}) \mid ...$$

Figure: Syntax of constraints

The forms of constraints:

- $\top, C_1 \wedge C_2$ — conjunction of constraints
- $Ind(\mathsf{T}, \mathsf{S})$ — values of type T contains elements of type S; i. e. given `x` : T and `idx` : $\mathbb{Z}$, `x [idx]` : S
- $Call(\mathsf{T}, \mathsf{S}_1, ..., \mathsf{S}_n, \mathsf{S})$ — values of type T callable with arguments of types S_i and result have type S; i. e. given `f` : T and $\mathtt{x}_i : \mathsf{S}_i$, $\mathtt{f}\ (\mathtt{x}_1,\ ...,\ \mathtt{x}_n) : \mathsf{S}$
- $Sexp_{\mathcal{X}}(\mathsf{T}, \mathsf{S}_1, ..., \mathsf{S}_n)$ — type T is S-expression type and one of it's constructors have form $\mathcal{X}\ (\mathsf{S}_1, ..., \mathsf{S}_n)$; i. e. given $\mathtt{x}_i : \mathsf{S}_i$, $\mathtt{X}\ (\mathtt{x}_1,\ ...,\ \mathtt{x}_n) : \mathsf{T}$
- ... — other constraints that we don't mention

Constraint System: Examples

- `fun (x) { x }` $: \forall \alpha.\ \top \Rightarrow (\alpha) \to \alpha$
- `fun (xs, i, x) { xs [i] := x ; xs }` $: \forall \alpha, \beta.\ Ind(\alpha, \beta) \Rightarrow (\alpha, \mathbb{Z}, \beta) \to \alpha$
- `fun (f, x) { f (x) }` $: \forall \alpha, \beta, \gamma.\ Call(\alpha, \beta, \gamma) \Rightarrow (\alpha, \beta) \to \gamma$
- `fun (x) { Some (x) }` $: \forall \alpha, \beta.\ Sexp_{Some}(\alpha, \beta) \Rightarrow (\beta) \to \alpha$

Constraint Solver

- We define binary relation "$C_1 \Vdash C_2$" that means "C_1 implies C_2", in terms of natural deduction
- The job of constraint solver is to suggest for the given "C" a type substitution "σ" so that "$\top \Vdash C\sigma$" satisfied, or state that it doesn't exist

! Any constraint "C" has form "$C_1 \wedge C_2 \wedge ... \wedge C_n$", where C_i is an **atomic constraint** (constraint without conjunctions)

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- So, we need a tool that could solve that CNF or state that there aren't solutions
- Here we meet **relational programming**

Relational Programming

Relational programming (especially, miniKanren):

- a special case of logic programming without side effects
- minimalistic embedded programming language (implementing as DSL library for existing programming languages)
- presented in The Reasoned Schemer [1]
- implemented for many popular programming languages: Scheme, OCaml, Kotlin, etc.
- have proven search completeness [2]

miniKanren: Terms

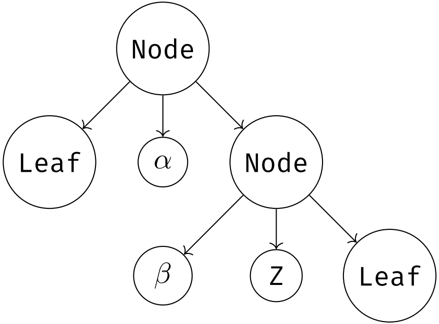
Term in logic programming is an expression of the following syntax:

$$t ::= \mathcal{X} \mid \mathcal{C}(t, ..., t),$$

where $\mathcal{X}$ — variable, $\mathcal{C}$ — tag

Examples of term:

- α
- `None`
- `Some`(α)
- `Z` — Peano number "0"
- `S(S(S(Z)))` — Peano number "3"
- `Node`$($`Leaf`$, \alpha,$ `Node`$(\beta,$ `Z`, `Leaf`$))$



miniKanren: Goals

Goal in miniKanren is a special case of logical expression:

$$g ::= \top \mid \bot \mid t = t \mid \mathcal{P}(t, ..., t) \mid g \wedge g \mid g \vee g \mid \exists \mathcal{X}.\ g,$$

where $\mathcal{P}$ — predicate, so we haven't implication, negation and forall quantifier

Examples of goal:

- $\alpha = \beta$
- $\alpha = \mathsf{A} \vee \alpha = \mathsf{B}$
- $P(\alpha) \wedge Q(\alpha, \beta)$
- $\alpha = \texttt{None} \vee \exists \beta.\ \alpha = \texttt{Some}(\beta)$
- $\exists \beta.\ \alpha = \mathsf{S}(\beta)$ — predicate "$\alpha \geq 1$" on Peano numbers

miniKanren: Definitions

Definition in miniKanren is like definition in logic programming but with a goal on the right-hand side:

$$\mathcal{P}(\mathcal{X}, ..., \mathcal{X}) \equiv g,$$

and the goal mustn't has free variables except listed on the left-hand side

Examples of definition:

- $P \equiv \top$
- $Q(\alpha, \beta) \equiv \alpha = \beta \vee \exists \gamma.\ \beta = \mathsf{S}(\gamma) \wedge Q(\alpha, \gamma)$
- $add^o(\alpha, \beta, \gamma) \equiv \alpha = \mathsf{Z} \wedge \beta = \gamma \vee \exists \alpha', \gamma'.\ \alpha = \mathsf{S}(\alpha') \wedge \gamma = \mathsf{S}(\gamma') \wedge add^o(\alpha', \beta, \gamma')$ — relation "$\alpha + \beta = \gamma$" on Peano numbers

miniKanren: Program

- miniKanren program is a list of relation definitions and a query (just a goal)
- Unlike logic programming languages like Prolog, definitions must be distinct by name
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  $add^o(\alpha, \beta, \gamma) \equiv \alpha = \mathsf{Z} \land \beta = \gamma \lor \exists \alpha', \gamma'.\ \alpha = \mathsf{S}(\alpha') \land \gamma = \mathsf{S}(\gamma') \land add^o(\alpha', \beta, \gamma')$

Query: $add^o(\xi, \psi, \mathsf{S}(\mathsf{S}(\mathsf{S}(\mathsf{S}(\mathsf{Z})))))$

miniKanren: Program

- miniKanren program is a list of relation definitions and a query (just a goal)
- Unlike logic programming languages like Prolog, definitions must be distinct by name
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Verifier-to-Solver Approach

Given problem:

- We can implement relational program "$P(x)$" that verifies problem solution "x" — verifier
- Running relational verifier "$P(\xi)$" on free variable "ξ" gives solution "$\xi \mapsto x$" — solver
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- Moreover, it gives the all possible solutions due to search completeness
- Unfortunately, relational program evaluation may hangs in practice due to exponential search complexity
- ...or simply, due to specific problem undecidability

Verifier-to-Solver: Application

- We may implement relational verifier for "$\Vdash$"
- "$\Vdash$" implemented directly in miniKanren works slowly and don't answer at all when proper substitution isn't exists
- Also, vanilla miniKanren implementations doesn't allow to deal with recursive terms, that are needed to deal with recursive types
- Wildcard logic variables [3]:
  - Most of miniKanren implementations supports inequality as a primitive
  - Wildcard variables allows to say "$\forall\psi.\ \xi \neq \mathtt{Cons}(\psi)$" instead of "$\exists\psi.\ \xi \neq \mathtt{Cons}(\psi)$"
- Non-relational optimizations to specialize relational implementation in the constraint solving problem:
  - Term shape check — non-relational primitives that give an ability to introspect current evaluation state and direct evaluator manually
  - Occurs hooks — an ability to hook an occurs check to permit unnatural recursive equations solving while unification

Term Shape Check

```
let rec contains x xs = ocanren {
  fresh x', xs' in xs == x'::xs' &
    { x == x'
    | x =/= x' & contains x xs'
    }
}
```

```
let rec contains x xs = ocanren
  { is_var xs &
    { fresh xs' in xs == x::xs' }
  | is_not_var xs &
    fresh x', xs' in xs == x'::xs' &
      { x == x'
      | x =/= x' & contains x xs'
      }
  }
```

- ? How to implement `contains`^{o} for sets encoded as lists?
- 👎 Trivial implementation will generate a lot of syntactically different, but semantically identical lists
- 👍 With the shape checking we are able to enforce a single solution in the case when the tail of list is free

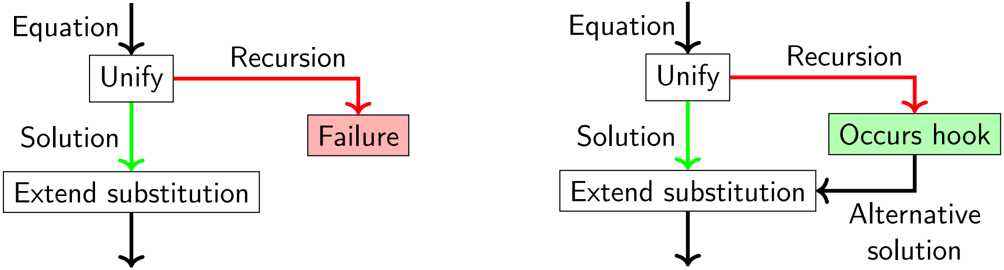
Search Space of Function Types

- ? How to solve a constraint of the form "$Call(\mathsf{T}, \mathsf{S}_1, \ldots, \mathsf{S}_n, \mathsf{S})$" when "T" is a free logic variable?
- The only form that "T" could be is: $\forall \mathcal{X}_1, \ldots, \mathcal{X}_m.\ C \Rightarrow (\mathsf{T}_1, \ldots, \mathsf{T}_n) \to \mathsf{T}'$
- In the implementation: `TArrow (fxs, fc, fts, ft)`
- A straightforward implementation will generate the variety of function types with all possible values of "`fxs`" and "`fc`"
- ! **Assumption:** all needed function types come from relational query, i. e. we don't need to generate them
- Just enforce the simplest possible type in the case when "T" is free: $\forall.\ \top \Rightarrow (\mathsf{S}_1, \ldots, \mathsf{S}_n) \to \mathsf{S}$
- ! This approach may cut off some solutions when "$Call$" is being solved too early

Solving Ordering

- Solving constraints left-to-right works poorly
- Let's support all planned to solve constraints and pick them one-by-one in a *good* order
- As a *good* order, we use picking a minimum by relational heuristic comparator, that inspects shapes (using "`is_var`") of constraint arguments
- For example, constraints of form "$Call(\mathsf{T}, \mathsf{S}_1, ..., \mathsf{S}_n, \mathsf{S})$" with logic variable in place of "T" are being picked last
- In addition to performance gain, we have fixed a problem with early solution that was mentioned before

Occurs Hooks



(a) Without occurs hooks (b) With occurs hooks

- Occurs hook is a callback that called when a recursive equation occurred in unification: $\xi = Term(\xi)$
- A hook returns an alternative right side of equation, so the new equation will be: $\xi = hook(\xi, Term(\xi))$

Evaluation

- We used the $\lambda^a\mathcal{M}^a$ compiler tests
- 👍 The majority of tests were successfully typechecked
- 👎 S-expression types are slowing down performance exponentionally of the number of $Sexp$ constraints
- 👍 Other constraints aren't so slow
- ! Occurs hooks gives an ability to deal with recursive types, but results not as good as possible, e.g.:
  - Good type: $\mu\alpha.\ \texttt{Nil} \sqcup \texttt{Cons}(\mathbb{Z}, \alpha)$
  - Produced type: $\texttt{Nil} \sqcup \texttt{Cons}(\mathbb{Z}, \mu\alpha.\ \texttt{Nil} \sqcup \texttt{Cons}(\mathbb{Z}, \alpha))$

Evaluation

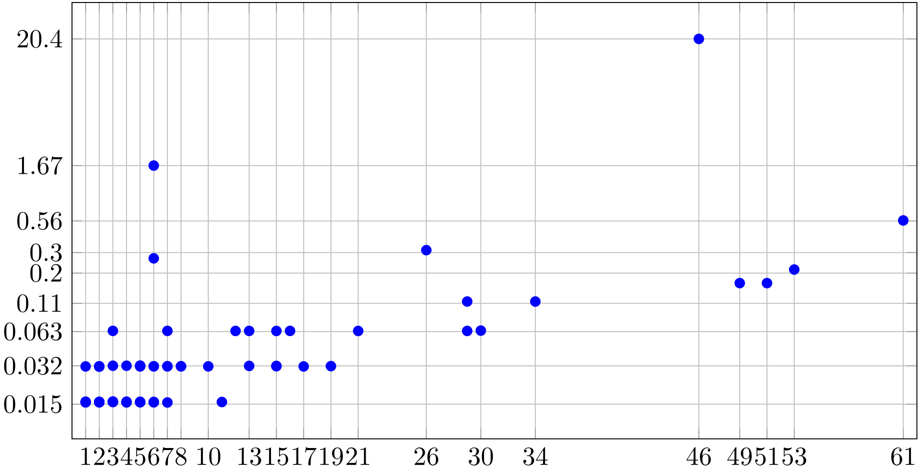
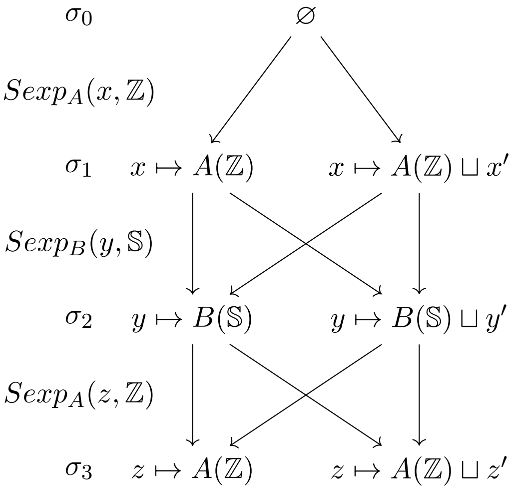


Figure: Elapsed time by the number of constraints

Evaluation: S-expression Types

- Example:
  $Sexp_A(x, \mathbb{Z}) \wedge Sexp_B(y, \mathbb{S}) \wedge Sexp_A(z, \mathbb{Z})$
- We have $m = 3$ different S-expression types x, y, z and $n = 2$ different constructors $A(\mathbb{Z}), B(\mathbb{S})$
- Number of branches is $\mathcal{O}(n^m)$
- As a result, we have about 8 branches only from this constraints
- It explains the high time consumption on the previous slide



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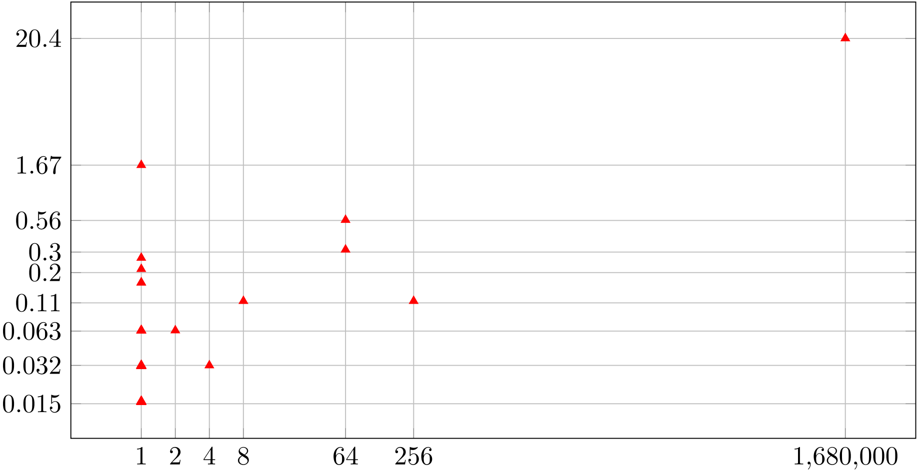

Figure: Elapsed time by the number of $Sexp$ branches

Results

- ✔ Relational programming is applicable to constraint-based type inference
- ✔ To specialize relational solvers we may inspect current state (`is_var`/`is_not_var`)
- ✔ Recursive terms may be partially emulated using non-relational hooks over "occurs check"
- This results are preliminary that need more research

More technical details are available in [4]

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Questions?

More technical details are available in [4]

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