On the Complexity of Semantic Integration of OWL Ontologies

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Abstract. We propose a new mechanism for integration of OWL ontologies using semantic import relations. In contrast to the standard OWL importing, we do not require all axioms of the imported ontologies to be taken into account for reasoning tasks, but only their logical implications over a chosen signature. This property comes natural in many ontology integration scenarios. In this paper, we study the complexity of reasoning over ontologies with semantic import relations and establish a range of tight complexity bounds for various fragments of OWL.

1 Introduction and Motivation

Ontology integration is a research topic which, in particular, aims at organizing information on different domains in a modular way so that information from one ontology can be reused in other ontologies. There is a number of approaches known in the literature [1–5, 8], which address this problem from the point of view of ontology linking and importing, see, e.g., an overview in [6]. Integration of multiple ontologies in OWL is organized via *importing*: an OWL ontology can refer to one or several other OWL ontologies, whose axioms must be implicitly present in the ontology. The importing mechanism is simple in that reasoning over an ontology with an import declaration is reduced to reasoning over the *import closure* consisting of the axioms of the ontology plus the axioms of the ontologies that are imported (possibly indirectly). For example, if ontology $\mathcal{O}_1$ imports ontologies $\mathcal{O}_2$ and $\mathcal{O}_3$, each of which, in turn, imports ontology $\mathcal{O}_4$, then the import closure of $\mathcal{O}_1$ consists of all axioms of $\mathcal{O}_1 - \mathcal{O}_4$. Although the OWL importing mechanism may work well for simple ontology integration scenarios, it may cause some undesirable side effects if used in complex import situations. To illustrate the problem, suppose that in the above example, $\mathcal{O}_4$ is an ontology describing a typical university. It may include concepts such as *Professor*, *Course*, and axioms stating, e.g., that each professor must teach some course: $\mathit{Professor} \sqsubseteq \exists \mathit{teaches}.\mathit{Course}$. Now suppose that $\mathcal{O}_2$ and $\mathcal{O}_3$ are ontologies describing respectively, Oxford and Cambridge universities that use $\mathcal{O}_4$ as a prototype. For example, $\mathcal{O}_2$ may include *mapping axioms* $\mathit{OxfordProfessor} \equiv \mathit{Professor}$, $\mathit{OxfordCourse} \equiv \mathit{Course}$, from which, due to the axiom in $\mathcal{O}_4$, it is now possible to conclude that $\mathit{OxfordProfessor} \sqsubseteq \exists \mathit{teaches}.\mathit{OxfordCourse}$. Likewise, using similar mapping axioms in $\mathcal{O}_3$, it is possible to obtain that $\mathit{CambridgeProfessor} \sqsubseteq \exists \mathit{teaches}.\mathit{CambridgeCourse}$. Finally, suppose that $\mathcal{O}_1$ is an ontology aggregating information about UK universities, importing, among others, the ontologies $\mathcal{O}_2$ and $\mathcal{O}_3$ for Oxford and Cambridge universities. Although the described scenario seems plausible, there will be some undesirable consequences in $\mathcal{O}_1$ due to the mapping axioms of $\mathcal{O}_2$ and $\mathcal{O}_3$ occurring in the import closure: $\mathit{OxfordProfessor} \equiv \mathit{CambridgeProfessor}$, $\mathit{OxfordCourse} \equiv \mathit{CambridgeCourse}$.

The main reason for these consequences is that the ontologies $\mathcal{O}_2$ and $\mathcal{O}_3$ happen to reuse the same ontology $\mathcal{O}_4$ in two different and incompatible ways. Had they instead used two different 'copies' of $\mathcal{O}_4$ as prototypes (with concepts renamed apart), no such problem would take place. Arguably, the primary purpose of $\mathcal{O}_2$ and $\mathcal{O}_3$ is to provide semantic description of the vocabulary for Oxford and Cambridge universities, and the means of how it is achieved—either by writing the axioms directly or reusing third party ontologies such as $\mathcal{O}_4$—should be an internal matter of these two ontologies and should not be exposed to the ontologies that import them. Motivated by the described scenario, in this paper we consider a refined mechanism for importing of OWL ontologies called *semantic importing*. The main difference with the standard OWL importing, is that each import is limited only to a subset of symbols. Intuitively, only logical properties of these symbols entailed by the imported ontology should be imported. These symbols can be regarded as the public (or external) vocabulary of the imported ontologies. For example, ontology $\mathcal{O}_2$ may declare the symbols *OxfordProfessor*, *OxfordCourse*, and *teaches* public, leaving the remaining symbols only for the internal use.

The main results of this paper are tight complexity bounds for reasoning over ontologies with semantic imports. We consider ontologies formulated in different fragments of OWL starting from the Description Logic (DL) $\mathcal{EL}$ and concluding with the DL $\mathcal{SROIQ}$, which corresponds to OWL 2. We also distinguish the case of acyclic imports, when ontologies cannot (possibly indirectly) import themselves. Our completeness results for ranges of DLs are summarized in the following table, where $\mathfrak{a}$ and $\mathfrak{c}$ denote the case of acyclic/cyclic imports:

DLs	Completeness	Theorems
$\mathcal{EL} - \mathcal{EL}^{++}$	ExpTime $\mathfrak{a}$	1, 8
containing $\mathcal{EL}$	RE (undecidable) $\mathfrak{c}$	2, 9
$\mathcal{ALC} - \mathcal{SHIQ}$	2ExpTime $\mathfrak{a}$	3, 8
$\mathcal{R} - \mathcal{SRIQ}$	3ExpTime $\mathfrak{a}$	4, 8
$\mathcal{ALCHOIF} - \mathcal{SHOIQ}$	coN2ExpTime $\mathfrak{a}$	5, 8
$\mathcal{ROIF} - \mathcal{SROIQ}$	coN3ExpTime $\mathfrak{a}$	6, 8

An extended version of the paper containing all proofs and additional materials can be downloaded from http://www.iis.nsk.su/persons/ponom/papers/

2 Preliminaries

We assume that the reader is familiar with the family of Description Logics from $\mathcal{EL}$ to $\mathcal{SROIQ}$, for which the syntax is defined using a recursively enumerable alphabet consisting of infinite disjoint sets $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_i}$ of *concept names* (or *primitive concepts*), *roles*, and *nominals*, respectively. The semantics of DLs is given by means of (first-order) interpretations. An ontology is a set of concept inclusions which are called ontology *axioms*. A *signature* is a subset of $\mathsf{N_C} \cup \mathsf{N_R} \cup \mathsf{N_i}$. Interpretations $\mathcal{I}$ and $\mathcal{J}$ are said to *agree* on a signature Σ, written as $\mathcal{I} =_\Sigma \mathcal{J}$, if the domains of $\mathcal{I}$ and $\mathcal{J}$ coincide and the interpretation of Σ-symbols in $\mathcal{I}$ is the same as in $\mathcal{J}$. We denote the reduct of an interpretation $\mathcal{I}$ onto a signature Σ as $\mathcal{I}|_\Sigma$. The signature of a concept C, denoted as $\mathtt{sig}\,(C)$, is the set of all concept names, roles, and nominals occurring in C. The signature of a concept inclusion or an ontology is defined identically.

Given a signature Σ, suppose one wants to import into an ontology $\mathcal{O}_1$ the semantics of Σ-symbols defined by some other ontology $\mathcal{O}_2$, while ignoring the rest of the symbols from $\mathcal{O}_2$. Intuitively, importing the semantics of Σ-symbols means reducing the class of models of $\mathcal{O}_1$ by removing those models that violate the restrictions on interpretation of these symbols, which are imposed by the axioms of $\mathcal{O}_2$:

Definition 1. *A* (semantic) import relation *is a tuple* $\pi = \langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle$ *where* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are ontologies and* Σ *a signature. In this case, we say that* $\mathcal{O}_1$ imports Σ from $\mathcal{O}_2$. *We say that a model* $\mathcal{I} \models \mathcal{O}_1$ *satisfies the import relation* π *if there exists a model* $\mathcal{J} \models \mathcal{O}_2$ *such that* $\mathcal{I} =_\Sigma \mathcal{J}$.

Example 1. Consider the import relation $\pi = \langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle$, with $\mathcal{O}_1 = \{B \sqsubseteq C\}$, $\mathcal{O}_2 = \{A \sqsubseteq \exists r.B,\ \exists r.C \sqsubseteq D\}$, and $\Sigma = \{A, B, C, D\}$. It can be easily shown using Definition 1 that a model $\mathcal{I} \models \mathcal{O}_1$ satisfies π if and only if $\mathcal{I} \models A \sqsubseteq D$.

Note that if Σ contains all symbols in $\mathcal{O}_2$ then $\mathcal{I} \models \mathcal{O}_1$ satisfies $\pi = \langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle$ if and only if $\mathcal{I} \models \mathcal{O}_1 \cup \mathcal{O}_2$. That is, the standard OWL import relation is a special case of the semantic import relation, when the signature contains all the symbols from the imported ontology. If $\mathcal{O}$ has several import relations $\phi_i = \langle \mathcal{O}, \Sigma_i, \mathcal{O}_i \rangle$, $(1 \le i \le n)$, one can define the entailment from $\mathcal{O}$ by considering only those models of $\mathcal{O}$ that satisfy all imports: $\mathcal{O} \models \alpha$ if $\mathcal{I} \models \alpha$ for every $\mathcal{I} \models \mathcal{O}$ which satisfies all $\pi_1, \ldots, \pi_n$. In practice, however, import relations can be nested: imported ontologies can themselves import other ontologies and so on. The following definition generalizes entailment to such situations.

Definition 2. *An* ontology network *is a finite set* $\mathcal{N}$ *of import relations between ontologies. For a DL* $\mathcal{L}$, *a* $\mathcal{L}$-ontology network *is a network, in which every ontology is a set of* $\mathcal{L}$*-axioms. A* model agreement *for* $\mathcal{N}$ *(over a domain* Δ*) is a mapping* μ *that assigns to every ontology* $\mathcal{O}$ *occurring in* $\mathcal{N}$ *a class* $\mu(\mathcal{O})$ *of models of* $\mathcal{O}$ *with domain* Δ *such that for every* $\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle \in \mathcal{N}$ *and every* $\mathcal{I}_1 \in \mu(\mathcal{O}_1)$ *there exists* $\mathcal{I}_2 \in \mu(\mathcal{O}_2)$ *such that* $\mathcal{I}_1 =_\Sigma \mathcal{I}_2$. *An interpretation* $\mathcal{I}$ *is a model of* $\mathcal{O}$ *in the network* $\mathcal{N}$ *(notation* $\mathcal{I} \models_\mathcal{N} \mathcal{O}$*) if there exists a model agreement* μ *for* $\mathcal{N}$ *such that* $\mathcal{I} \in \mu(\mathcal{O})$. *An ontology* $\mathcal{O}$ *entails a concept inclusion* φ *in the network* $\mathcal{N}$ *(notation* $\mathcal{O} \models_\mathcal{N} \varphi$*) if* $\mathcal{I} \models \varphi$, *whenever* $\mathcal{I} \models_\mathcal{N} \mathcal{O}$.

An ontology network can be seen as a labeled directed multigraph in which nodes are labeled by ontologies and edges are labeled by sets of signature symbols. Note that Definition 2 also allows for *cyclic networks* if this graph is cyclic. Note that if $\mathcal{O} \models \varphi$ then $\mathcal{O} \models_\mathcal{N} \varphi$, for every network $\mathcal{N}$.

In this paper, we are concerned with the complexity of *entailment in ontology networks*, that is, given a network $\mathcal{N}$, an ontology $\mathcal{O}$ and an axiom φ, decide whether $\mathcal{O} \models_\mathcal{N} \varphi$. We study the complexity of this problem wrt the *size* of an ontology network $\mathcal{N}$, which is defined as the total length of axioms (considered as strings) occurring in ontologies from $\mathcal{N}$.

3 Expresiveness of Ontology Networks

First, we illustrate the expressiveness of ontology networks by showing that acyclic networks allow for succinctly representing axioms with nested concepts and role chains of exponential size, while cyclic ones allow for succinctly representing infinite sets of axioms of a special form. For a natural number $n \geqslant 0$, let $\exists(r, C)^n.D$ be a shortcut for the nested concept $\exists r.(C \sqcap \exists r.(C \sqcap \ldots \text{n times} \ldots \sqcap \exists r.(C \sqcap D)\ldots))$, where C, D are DL concepts and r a role (in case $n = 0$ the above concept is set to be D). For $n \geqslant 1$, let $(r)^n$ denote the role chain $r \circ \ldots \text{n times} \ldots \circ r$. For a given $n \geqslant 0$, let $\mathtt{1exp}(n)$ be the notation for 2^n and for $k \geqslant 1$, let $(\mathtt{k+1})\mathtt{exp}(n) = 2^{\mathtt{kexp}(n)}$. Then $\exists(r, C)^{\mathtt{kexp}(n)}.D$ (respectively, $(r)^{\mathtt{kexp}(n)}$) stands for a nested concept (role chain) of the form above having size exponential in n. In the following, we use abbreviations $\exists(r, C)^n := \exists(r, C)^n.\top$ and $\exists r^n.C := \exists(r, \top)^n.C$. For $n \geqslant 1$, the expression $\forall r^n.C$ will be used as a shortcut for $\neg\exists r.\exists r^{n-1}.\neg C$ and for $n \geqslant 2$, $\forall r^{<n}.C$ will stand for $\bigsqcap_{1 \leqslant m < n} \forall r^m.C$. Let $\mathcal{O}$ be an ontology and $\mathcal{N}$ an ontology network. $\mathcal{O}$ is said to be *expressible* by $\mathcal{N}$ if there is an ontology $\mathcal{O}_\mathcal{N}$ in $\mathcal{N}$ such that $\{\mathcal{I}|_{\mathtt{sig}(\mathcal{O})} \mid \mathcal{I} \models_\mathcal{N} \mathcal{O}_\mathcal{N}\} = \{\mathcal{J}|_{\mathtt{sig}(\mathcal{O})} \mid \mathcal{J} \models \mathcal{O}\}$. In case we want to stress the role of ontology $\mathcal{O}_\mathcal{N}$ in the network $\mathcal{N}$, we say that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-*expressible*. An axiom φ is expressible by a network $\mathcal{N}$ if so is ontology $\mathcal{O} = \{\varphi\}$. The next two lemmas follow immediately from the definition of expressibility.

Lemma 1. *Every ontology $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O})$-expressible, where $\mathcal{N}$ is the network consisting of the single import relation $\langle \mathcal{O}, \varnothing, \varnothing \rangle$. If an ontology $\mathcal{O}_i$ is $(\mathcal{N}_i, \mathcal{O}'_i)$-expressible, for a network $\mathcal{N}_i$, ontology $\mathcal{O}'_i$, and $i = 1, 2$, then $\mathcal{O}_1 \cup \mathcal{O}_2$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible, for ontology $\mathcal{O}_\mathcal{N} = \varnothing$ and a network*[1] $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \{\langle \mathcal{O}_\mathcal{N}, \mathtt{sig}(\mathcal{O}_i), \mathcal{O}'_i \rangle\}_{i=1,2}$.

For an axiom φ and a set of concepts $\{C_1, \ldots, C_n\}$, $n \geqslant 1$, let us denote by $\varphi[C_1 \mapsto D_1, \ldots, C_n \mapsto D_n]$ the axiom obtained by substituting every concept C_i with a concept D_i in φ. For an ontology $\mathcal{O}$, let $\mathcal{O}[C_1 \mapsto D_1, \ldots, C_n \mapsto D_n]$ be a notation for $\bigcup_{\varphi \in \mathcal{O}} \varphi[C_1 \mapsto D_1, \ldots, C_n \mapsto D_n]$.

Lemma 2. *Let $\mathcal{L}$ be a DL and $\mathcal{O}$ an ontology, which is expressible by a $\mathcal{L}$-ontology network $\mathcal{N}$. Let $C_1, \ldots, C_n$ be $\mathcal{L}$-concepts and $\{A_1, \ldots, A_n\}$ a set of concept names such that $A_i \in \mathtt{sig}(\mathcal{O})$, for $i = 1, \ldots, n$ and $n \geqslant 1$. Then ontology $\tilde{\mathcal{O}} = \mathcal{O}[A_1 \mapsto C_1, \ldots, A_n \mapsto C_n]$ is expressible by a $\mathcal{L}$-ontology network, which is acyclic if so is $\mathcal{N}$ and has size polynomial in the size of $\mathcal{N}$ and C_i, $i = 1, \ldots, n$.*

Proof Sketch. Denote $\Sigma = \{A_1, \ldots, A_n\}$ and let $\Sigma' = \{A'_1, \ldots, A'_n\}$ be a set of fresh concept names. Consider ontology $\mathcal{O}' = \mathcal{O} \cup \{A_i \equiv A'_i\}_{i=1,\ldots,n}$. By Lemma 1, $\mathcal{O}'$ is $(\mathcal{N}', \mathcal{O}_{\mathcal{N}'})$-expressible, for an ontology $\mathcal{O}_{\mathcal{N}'}$ and an acyclic $\mathcal{L}$-ontology network $\mathcal{N}'$ having a linear size (in the size of $\mathcal{N}$). Consider ontology network $\mathcal{N}'' = \langle \mathcal{O}_{\mathcal{N}''}, (\mathtt{sig}(\mathcal{O}') \setminus \Sigma) \cup \Sigma', \mathcal{O}_{\mathcal{N}'} \rangle$, where $\mathcal{O}_{\mathcal{N}''} = \varnothing$. Then obviously, ontology $\mathcal{O}'' = \mathcal{O}[A_1 \mapsto A'_1, \ldots, A_n \mapsto A'_n]$ is $(\mathcal{N}'', \mathcal{O}_{\mathcal{N}''})$-expressible. Similarly, by Lemma 1, ontology $\mathcal{O}''_C = \mathcal{O}'' \cup \{A'_i \equiv C_i\}_{i=1,\ldots,n}$ is $(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$-expressible, for an ontology $\mathcal{O}_{\tilde{\mathcal{N}}}$ and an acyclic $\mathcal{L}$-ontology network $\tilde{\mathcal{N}}$ having a linear size (in the size of $\mathcal{N}$ and C_i, for $i = 1, \ldots, n$). Clearly, it holds $\mathcal{O}_{\tilde{\mathcal{N}}} \models_{\tilde{\mathcal{N}}} \tilde{\mathcal{O}}$ and it is easy to show that $\tilde{\mathcal{O}}$ is $(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$-expressible.

Now we formulate the results on the expressiveness of acyclic $\mathcal{EL}$- and $\mathcal{ALC}$-ontology networks.

Lemma 3. *An axiom φ of the form $Z \equiv \exists(r, A)^{\mathtt{1exp}(n)}.B$, where $Z, A, B \in \mathsf{N_C}$, and $n \geqslant 0$, is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in n.*

[1] Note that the size of $\mathcal{N}$ is linear in the sizes of $\mathcal{N}_1$ and $\mathcal{N}_2$.

Proof Sketch. We show by induction on n that there exists an acyclic $\mathcal{EL}$-ontology network $\mathcal{N}_n$ and ontology $\mathcal{O}_n$ such that φ is $(\mathcal{N}_n, \mathcal{O}_n)$-expressible. For $n = 0$, we define $\mathcal{N}_0 = \{\langle \mathcal{O}_0, \varnothing, \varnothing \rangle\}$ and $\mathcal{O}_0 = \{Z \equiv \exists r.(A \sqcap B)\}$. In the induction step, let $\{Z \equiv \exists (r, A)^{\mathtt{1exp}(n-1)}.B\}$ be $(\mathcal{N}_{n-1}, \mathcal{O}_{n-1})$-expressible, for $n \geqslant 1$. Consider ontologies $\mathcal{O}^1_{copy} = \{B \equiv U\}$, $\mathcal{O}^2_{copy} = \{U \equiv Z\}$, $\mathcal{O}_n = \varnothing$. Let $\mathcal{N}_n$ be the union of $\mathcal{N}_{n-1}$ with the set of the following import relations: $\langle \mathcal{O}^i_{copy}, \{Z, A, B, r\}, \mathcal{O}_{n-1} \rangle$, for $i = 1, 2$, $\langle \mathcal{O}_n, \{Z, A, U, r\}, \mathcal{O}^1_{copy} \rangle$, and $\langle \mathcal{O}_n, \{U, A, B, r\}, \mathcal{O}^2_{copy} \rangle$. Let us verify that $\{\mathcal{I}|_{\mathtt{sig}(\varphi)} \mid \mathcal{I} \models_{\mathcal{N}_n} \mathcal{O}_n\} = \{\mathcal{I}|_{\mathtt{sig}(\varphi)} \mid \mathcal{I} \models \varphi\}$. By the induction assumption we have $\mathcal{O}_{n-1} \models_{\mathcal{N}_n} Z \equiv \exists (r, A)^{\mathtt{1exp}(n-1)}.B$. Then by the definition of $\mathcal{N}_n$, it holds $\mathcal{O}^1_{copy} \models_{\mathcal{N}_n} Z \equiv \exists (r, A)^{\mathtt{1exp}(n-1)}.U$ and $\mathcal{O}^2_{copy} \models_{\mathcal{N}_n} U \equiv \exists (r, A)^{\mathtt{1exp}(n-1)}.B$ and thus, $\mathcal{O}_n \models_{\mathcal{N}_n} \{Z \equiv \exists (r, A)^{\mathtt{1exp}(n-1)}.U,\ U \equiv \exists (r, A)^{\mathtt{1exp}(n-1)}.B\}$, which yields $\mathcal{O}_n \models_{\mathcal{N}_n} \varphi$. To complete the proof, we show that any model $\mathcal{I} \models \varphi$ can be expanded to a model $\mathcal{J}_n \models_{\mathcal{N}_n} \mathcal{O}_n$ by setting $U^{\mathcal{J}_n} = (\exists (r, A)^{\mathtt{1exp}(n-1)}.B)^{\mathcal{I}}$.

The following three claims are proved identically to Lemma 3:

Lemma 4. *An axiom of the form $Z \sqsubseteq \exists (r, A)^{\mathtt{1exp}(n)}.B$, where $Z, A, B \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in n.*

Lemma 5. *An axiom of the form $Z \equiv \forall r^{\mathtt{1exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{ALC}$-ontology network of size polynomial in n.*

Lemma 6. *An axiom φ of the form $(r)^{\mathtt{2exp}(n)} \sqsubseteq s$, where r, s are roles and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

We continue with results on the expressiveness of acyclic $\mathcal{R}$-ontology networks.

Lemma 7. *An axiom φ of the form $Z \sqsubseteq \forall r^{\mathtt{2exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. Consider ontology $\mathcal{O} = \{Z \sqsubseteq \forall s.A,\quad (r)^{\mathtt{2exp}(n)} \sqsubseteq s\}$. Clearly, it holds $\mathcal{O} \models \varphi$ and any model $\mathcal{I} \models \varphi$ can be expanded to a model $\mathcal{J} \models \mathcal{O}$ by setting $s^{\mathcal{J}} = ((r)^{\mathtt{2exp}(n)})^{\mathcal{I}}$. By Lemmas 1, 6, $\mathcal{O}$ is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n, from which the claim follows.

Similarly, we demonstrate that an axiom of the form $Z \sqsubseteq \forall r^{<\mathtt{2exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n and show the next statement, which is an analogue of Lemma 4 for the case of double exponent.

Lemma 8. *An axiom φ of the form $Z \sqsubseteq \exists (r, A)^{\mathtt{2exp}(n)}.B$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. Consider ontology $\bar{\mathcal{O}} = \{Z \sqsubseteq \exists s.\top,\ \ Z \sqsubseteq \forall s^{<\mathtt{2exp}(n)}.X,\ \ Z \sqsubseteq \forall s^{\mathtt{2exp}(n)}.Y,$ $s \sqsubseteq r\}$. By Lemmas 1, 7, and the claim above, we show that $\bar{\mathcal{O}}$ is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n. Then by Lemma 2, so is ontology $\mathcal{O} = \bar{\mathcal{O}}[X \mapsto A \sqcap \exists s.\top,\ Y \mapsto A \sqcap B]$. Clearly, we have $\mathcal{O} \models \varphi$. Now let $\mathcal{I}$ be an arbitrary model of φ and for $m = \mathtt{2exp}(n)$, let $x_0, \ldots, x_m$ be arbitrary domain elements such that $x_0 \in Z^{\mathcal{I}}$, $\langle x_0, x_1 \rangle \in r^{\mathcal{I}}$, and $\langle x_i, x_{i+1} \rangle \in r^{\mathcal{I}}$, $x_i \in A^{\mathcal{I}}$, for $1 \leqslant i < m$, and $x_m \in A^{\mathcal{I}} \sqcap B^{\mathcal{I}}$. Let $\mathcal{J}$ be an expansion of $\mathcal{I}$ in which $s^{\mathcal{J}} = \{\langle x_i, x_{i+1} \rangle\}_{0 \leqslant i < m}$. Then we have $\mathcal{J} \models \mathcal{O}$, from which the claim follows.

Lemma 9. *An axiom φ of the form $Z \equiv \forall r^{\mathtt{2exp}(n)}.A$, where $Z, A \in \mathsf{N_C}$ and $n \geqslant 0$, is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n.*

Proof. Consider ontology $\bar{\mathcal{O}} = \{Z \sqsubseteq \forall r^{\mathtt{2exp}(n)}.A, \quad \bar{Z} \sqsubseteq \exists r^{\mathtt{2exp}(n)}.\bar{A}\}$. By Lemmas 1, 7, 8, $\bar{\mathcal{O}}$ is expressible by an acyclic $\mathcal{R}$-ontology network of size polynomial in n and by Lemma 2, so is ontology $\mathcal{O} = \bar{\mathcal{O}}[\bar{Z} \mapsto \neg Z, \ \bar{A} \mapsto \neg A]$. It remains to note that $\mathcal{O}$ and $\{\varphi\}$ are equivalent, so the claim is proved.

Lemma 10. *Let $\mathcal{L}$ be a DL and $\mathcal{O}$ an ontology, which is expressible by a $\mathcal{L}$-ontology network $\mathcal{N}$. Let $C_1, \ldots, C_m$ be $\mathcal{L}$-concepts and $\{A_1, \ldots, A_m\}$ a set of concept names such that $A_i \in \mathtt{sig}(\mathcal{O})$, for $i = 1, \ldots, m$ and $m \geqslant 1$. Then for $k = 1, 2$ and $n \geqslant 0$, ontology $\tilde{\mathcal{O}} = \mathcal{O}[A_1 \mapsto \forall r^{\mathtt{kexp}(n)}.C_1, \ldots, A_m \mapsto \forall r^{\mathtt{kexp}(n)}.C_m]$ is expressible by a $\mathcal{L}'$-ontology network, which is acyclic if so is $\mathcal{N}$ and has size polynomial in the size of $\mathcal{N}$, n, and C_i, for $i = 1, \ldots, m$, where: $\mathcal{L}' = \mathcal{L}$ if $\mathcal{L}$ contains $\mathcal{ALC}$ and $k = 1$, or $\mathcal{L}' = \mathcal{L}$ if $\mathcal{L}$ contains $\mathcal{R}$ and $k = 2$.*

Proof. The proof uses Lemmas 5, 9 and is identical to the proof of Lemma 2.

The next statement can be shown similarly by using Lemma 3:

Lemma 11. *In the conditions of Lemma 10, for $n \geqslant 0$, ontology $\tilde{\mathcal{O}} = \mathcal{O}[A_1 \mapsto \exists r^{\mathtt{1exp}(n)}.C_1, \ldots, A_m \mapsto \exists r^{\mathtt{1exp}(n)}.C_m]$ is expressible by a $\mathcal{L}'$-ontology network, which is acyclic if so is $\mathcal{N}$ and has size polynomial in the size of $\mathcal{N}$, n, and C_i, for $i = 1, \ldots, m$, where $\mathcal{L}' = \mathcal{L}$ if $\mathcal{L}$ contains $\mathcal{EL}$.*

Lemma 12. *Let $\mathcal{L}$ be a DL and $\mathcal{O}$ an ontology, which is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible, for a $\mathcal{L}$-ontology network $\mathcal{N}$ and an ontology $\mathcal{O}_{\mathcal{N}}$. Let $\{A_1, \ldots, A_n\}$, $n \geqslant 1$, be concept names such that $A_i \in \mathtt{sig}(\mathcal{O})$, for $i = 1, \ldots, n$, and let $\{C_1, \ldots, C_n\}$ be $\mathcal{L}$-concepts, where every C_i is of the form $\exists (r, D)^p.A_i$, for some role r, concept name D, and $p \geqslant 1$. Then ontology $\tilde{\mathcal{O}} = \bigcup_{m \geqslant 0} \mathcal{O}_m$, where $\mathcal{O}_0 = \mathcal{O}$ and $\mathcal{O}_{m+1} = \mathcal{O}_m[A_1 \mapsto C_1, \ldots, A_n \mapsto C_n]$, for all $m \geqslant 0$, is expressible by a cyclic $\mathcal{L}$-ontology network.*

Proof Sketch. Let $\sigma = \{B_1, \ldots, B_k\} = \bigcup_{i=1,\ldots,n} (\mathtt{sig}(C_i) \cap \mathsf{N_C})$ and $\sigma' = \{B_1', \ldots, B_k'\}$ be a set of fresh concept names disjoint with σ and $\mathtt{sig}(\mathcal{O})$. Let $\{C_1', \ldots, C_n'\}$ be 'copy' concepts obtained from $C_1, \ldots, C_n$ by replacing every B_i with B_i', for $i = 1, \ldots, k$. Consider ontologies $\tilde{\mathcal{O}}_{\mathcal{N}'} = \{ \ B_i \equiv B_i' \ \}_{i=1,\ldots,k}$, $\mathcal{O}' = \{ \ A_i \equiv C_i' \ \}_{i=1,\ldots,n}$, and an ontology network $\mathcal{N}'$ given by the union of $\mathcal{N}$ with the set of import relations $\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \mathtt{sig}(\mathcal{O}), \mathcal{O}_{\mathcal{N}} \rangle$, $\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \Sigma', \mathcal{O}' \rangle$, $\langle \mathcal{O}', \Sigma, \tilde{\mathcal{O}}_{\mathcal{N}'} \rangle$, where $\Sigma' = (\Sigma \setminus \sigma) \cup \sigma'$ and $\Sigma = \mathtt{sig}(\mathcal{O}) \cup \bigcup_{i=1,\ldots,n} \mathtt{sig}(C_i)$. We claim that ontology $\tilde{\mathcal{O}}$ is $(\mathcal{N}', \tilde{\mathcal{O}}_{\mathcal{N}'})$-expressible. Denote $\tilde{\mathcal{O}}' = \bigcup_{m \geqslant 0} \mathcal{O}_m'$, where $\mathcal{O}_m' = \mathcal{O}_m[B_1 \mapsto B_1', \ldots, B_k \mapsto B_k']$, for all $m \geqslant 0$. First, we show by induction that $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_m$, for all $m \geqslant 0$. The induction base for $n = 0$ is trivial, since we have $\mathcal{O}_0 = \mathcal{O}$ and $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible, $\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \mathtt{sig}(\mathcal{O}), \mathcal{O}_{\mathcal{N}} \rangle \in \mathcal{N}'$, and thus, $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}$. Suppose $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_m$, for some $m \geqslant 0$. Since $\langle \mathcal{O}', \Sigma, \tilde{\mathcal{O}}_{\mathcal{N}'} \rangle \in \mathcal{N}'$, we have $\mathcal{O}' \models_{\mathcal{N}'} \mathcal{O}_m$ and thus, by the equivalences in $\mathcal{O}'$, it holds $\mathcal{O}' \models_{\mathcal{N}'} \mathcal{O}_{m+1}'$. Since $\langle \tilde{\mathcal{O}}_{\mathcal{N}'}, \Sigma', \mathcal{O}' \rangle$, we have $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_{m+1}'$ and hence, by the equivalences in $\tilde{\mathcal{O}}_{\mathcal{N}'}$, it holds that $\tilde{\mathcal{O}}_{\mathcal{N}'} \models_{\mathcal{N}'} \mathcal{O}_{m+1}$. Finally, in the full proof we show that any model $\mathcal{I}$ of $\tilde{\mathcal{O}}$ can be expanded to a model $\mathcal{J} \models_{\mathcal{N}'} \tilde{\mathcal{O}}_{\mathcal{N}'}$, which shows that $\tilde{\mathcal{O}}$ is $(\mathcal{N}', \tilde{\mathcal{O}}_{\mathcal{N}'})$-expressible.

4 Hardness Results

We use reductions from the word problem for Turing machines (TMs) and alternating Turing machines (ATMs) to obtain most of the hardness results. We define a Turing Machine (TM) as a tuple $M = \langle Q, \mathcal{A}, \delta \rangle$, with $\mathtt{q_h} \in Q$ being the accepting state, and assume w.l.o.g. that configuration of M is a word in the alphabet $Q \cup \mathcal{A}$. An initial configuration is a word of the form $\mathtt{b} \ldots \mathtt{b}\mathtt{q_0}\mathtt{b} \ldots \mathtt{b}$, where $\mathtt{q_0} \in Q$ and $\mathtt{b} \in \mathcal{A}$ is the blank symbol. It is a well-known property of the transition functions of Turing machines that the symbol c'_i at position i of a successor configuration $\mathfrak{c}'$ is uniquely determined by a 4-tuple of symbols $c_{i-2}, c_{i-1}, c_i, c_{i+1}$ at positions $i-2$, $i-1$, i, and $i+1$ of a configuration $\mathfrak{c}$. We assume that this correspondence is given by the (partial) function δ' and use the notation $c_{i-2}c_{i-1}c_ic_{i+1} \overset{\delta'}{\mapsto} c'_i$.

Theorem 1. *Entailment in acyclic $\mathcal{EL}$-ontology networks is* ExpTime*-hard.*

Proof Sketch. We reduce the word problem for TMs making exponentially many steps to entailment in $\mathcal{EL}$-ontology networks. Let M be a TM and $n = \mathtt{1exp}(m)$ an exponential, for $m \geqslant 0$. Consider an ontology $\mathcal{O}$ defined for M and n by the following axioms:

$$A \sqsubseteq \exists r^{n\cdot(2n+3)}.(\mathtt{q_0} \sqcap \exists(r, \mathtt{b})^{2n+2}), \quad \text{where } A \notin Q \cup \mathcal{A} \tag{1}$$

$$\exists r^{2n}(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W, \quad \text{for } XYUZ \overset{\delta'}{\mapsto} W \tag{2}$$

$$\exists r.\mathtt{q_h} \sqsubseteq H, \;\; \exists r.H \sqsubseteq H, \qquad \text{where } H \notin Q \cup \mathcal{A} \tag{3}$$

The first axiom gives a r-chain containing $n+1$ segments of length $2n+3$, which are used to store fragments of consequent configurations of M. We assume the following enumeration of segments in the r-chain: $\underbrace{\ldots}_{s_n} \;..\; \underbrace{\ldots}_{s_1}\underbrace{\mathtt{q_0}\mathtt{b}\ldots\mathtt{b}}_{s_0}$, i.e., s_0 represents a fragment of the initial configuration $\mathfrak{c}_0$ of M. For $0 \leqslant i < n$, every i-th and $(i+1)$-st segments in the r-chain are reserved for a pair of configurations $\mathfrak{c}_i, \mathfrak{c}_{i+1}$ such that $\mathfrak{c}_{i+1}$ is a successor of $\mathfrak{c}_i$. Axioms (2), with $X, Y, U, Z, W \in Q \cup \mathcal{A}$, represent transitions of M and define the 'content' of $(i+1)$-st segment based on the 'content' of i-th segment. Finally, axioms (3) are used to initialise the halting marker H and propagate it to the 'left' of the r-chain. We show that M accepts the empty word in n steps iff $\mathcal{O} \models A \sqsubseteq H$. For the 'only if' direction we assume there is a sequence of configurations $\mathfrak{c}_0, \ldots, \mathfrak{c}_n$ such that for all $0 \leqslant i < n$, $\mathfrak{c}_{i+1}$ is a successor configuration of $\mathfrak{c}_i$ and $\mathtt{q_h}$ is the state symbol in $\mathfrak{c}_n$. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and a domain element such that $a \in A^{\mathcal{I}}$. Then by axiom (1), there is an r-chain outgoing from x, which contains segments $s_0, \ldots, s_n$ of length $2n+3$, where s_0 represents a fragment of $\mathfrak{c}_0$. It can be shown by induction that due to axioms (2), every segment s_i represents a fragment of $\mathfrak{c}_i$, for $1 \leqslant i \leqslant n$, and contains the state symbol from $\mathfrak{c}_i$. Then by axiom (3), it follows that $a \in H^{\mathcal{I}}$. For the 'if' direction, one can provide a model $\mathcal{I}$ of $\mathcal{O}$ such that $A^{\mathcal{I}} = \{a\}$ is a singleton, ${\mathtt{q_h}}^{\mathcal{I}} = H^{\mathcal{I}} = \varnothing$, and $\mathcal{I}$ gives an r-chain outgoing from a, which contains $n+1$ segments representing fragments of consequent configurations of M, neither of which contains $\mathtt{q_h}$. To complete the proof of the theorem we show that ontology $\mathcal{O}$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. Consider axiom (1) and a concept inclusion φ of the form $A \sqsubseteq \exists r^{n\cdot(2n+3)}.B$, where B is a concept name. Observe that it is equivalent to $A \sqsubseteq \underbrace{\exists r^p.\exists r^p}_{\text{2 times}}.\underbrace{\exists r^n.\exists r^n.\exists r^n}_{\text{3 times}}.B$, where $p = \mathtt{1exp}(2m)$. Consider axiom ψ of the form $A \sqsubseteq B$. By iteratively applying Lemma 11 we obtain that $\psi[B \mapsto \exists r^p.\exists r^p.B]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. By repeating this argument we obtain the same for φ. Further, by Lemma 2, the axiom $\theta = \varphi[B \mapsto \mathtt{q_0} \sqcap B]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m. Again, by iteratively applying Lemma 11 together with Lemma 2 we conclude that $\theta[B \mapsto \exists(r, \mathtt{b})^{2n+2}]$ is expressible by an acyclic $\mathcal{EL}$-ontology network of size polynomial in m and thus, so is axiom (1). The expressibility of axioms (2) is shown identically. The remaining axioms of ontology $\mathcal{O}$ are $\mathcal{EL}$-axioms whose size does not depend on m. By applying Lemma 1 we obtain that there exists an acyclic $\mathcal{EL}$-ontology network $\mathcal{N}$ of size polynomial in m and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, it holds $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq H$ iff M accepts the empty word in $\mathtt{1exp}(m)$ steps.

Theorem 2. *Entailment in cyclic $\mathcal{EL}$-ontology networks is* **RE**-*hard.*

Proof Sketch. For a TM M, we define an infinite ontology $\mathcal{O}$, which contains variants of axioms (1-2) from Theorem 1 and additional axioms for a correct implementation of transitions of M:

$$A \sqsubseteq \exists r^k.(\exists v^l.L \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists(r, \mathtt{b})^{2l+2})), \qquad \text{where } A, L, \varepsilon \notin Q \cup \mathcal{A} \tag{4}$$

$$\exists r.\exists v^k.L \sqsubseteq \exists v^k.L, \quad k \geqslant 0 \tag{5}$$

$$\exists v^k.L \sqcap \exists r^{2k+4}.\varepsilon \sqsubseteq \varepsilon, \quad k \geqslant 0 \tag{6}$$

$$\exists v^k.L \sqcap \exists r^{2k+1}.(X \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z))) \sqsubseteq W, \qquad \text{for } XYUZ \overset{\delta'}{\mapsto} W \tag{7}$$

$$\exists v^k.L \sqcap \exists r^{2k}.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(Y \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W, \text{ for } \mathtt{b}YUZ \overset{\delta'}{\mapsto} W \tag{8}$$

$$\exists v^k.L \sqcap \exists r^{2k}.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.(U \sqcap \exists r.Z)))) \sqsubseteq W, \text{ for } \mathtt{bb}UZ \overset{\delta'}{\mapsto} W \tag{9}$$

$$\exists v^k.L \sqcap \exists r^{2k}.(X \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\mathtt{b} \sqcap \exists r.(\varepsilon \sqcap \exists r.Z)))) \sqsubseteq W, \text{ for } X\mathtt{bb}Z \overset{\delta'}{\mapsto} W \tag{10}$$

$$\mathtt{q_h} \sqsubseteq H, \ \exists r.H \sqsubseteq H, \ \text{where } H \notin Q \cup \mathcal{A} \tag{11}$$

Axioms (4) give an infinite family of r-chains, each having a 'prefix' of length $k+1$, for $k \geqslant 0$ (reserved for fragments of consequent configurations of M), and a 'postfix' containing a chain of length $2l+3$, for $l \geqslant 0$, which represents a fragment of the initial configuration $\mathfrak{c}_0$. Propagated to the 'left' by axioms (5), concept $\exists v^k.L$ indicates the length of the 'postfix' for $\mathfrak{c}_0$ on every r-chain given by axioms (4). The concept ε is used to separate fragments of consequent configurations of M and is propagated by axioms (6). The family of axioms, (7)-(10), with $X, Y, U, Z, W \in Q \cup \mathcal{A}$ and $k \geqslant 0$, implements transitions of M. Concept $\exists v^k.L$ guarantees that transitions have effect only along r-chains which represent a fragment of $\mathfrak{c}_0$ of length $2k+3$. Since $\varepsilon \notin Q \cup \mathcal{A}$, the transitions involving ε are implemented separately by axioms (8)-(10). The more involved implementation of transitions (in comparison to Theorem 1) allows to prevent defect situations, when there are two consequent segments s_i,s_{i+1} of an r-chain, which represent fragments of configurations $\mathfrak{c}_i$,$\mathfrak{c}_{i+1}$ of M, respectively, but $\mathfrak{c}_{i+1}$ is not a successor of $\mathfrak{c}_i$. In Theorem 1, the prefix of length $n \cdot (2n+3)$ given by axiom (1) guarantees a correct implementation of up to n transitions of the TM. The situation is different in the infinite case, since the prefix reserved for fragments of consequent configurations of M can be of any length, due to axioms (4).

We prove that M halts iff $\mathcal{O} \models A \sqsubseteq H$. Suppose that $\mathfrak{c}_0$ is an accepting configuration and M halts in n steps; w.l.o.g. we assume that $n > 1$. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and $a \in A^{\mathcal{I}}$ a domain element. Due to axioms (4), $\mathcal{I}$ is a model of the concept inclusion: $A \sqsubseteq \exists r^{n\cdot(2n+4)}.(\exists v^n.L \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists(r,\mathtt{b})^{2n+2}))$ and thus, $\mathcal{I}$ gives a r-chain containing $n+1$ segments of length $2n+3$ separated by ε. By using arguments from the proof of Theorem 1, it can be shown that due to axioms (5) - (10), these segments represent fragments of consequent configurations of M, starting with $\mathfrak{c}_0$, and there is an element b in the r-chain such that $b \in \mathtt{q_h}^{\mathcal{I}}$. Then by axiom (11), it holds $a \in H^{\mathcal{I}}$. For the 'if' direction, one can show that if M does not halt, then there exists a model $\mathcal{I}$ of $\mathcal{O}$ such that $\mathtt{q_h}^{\mathcal{I}} = H^{\mathcal{I}} = \varnothing$, $A^{\mathcal{I}} = \{a\}$ is a singleton and there are infinitely many disjoint r-chains $\{R_{m,n}\}_{m,n\geqslant 1}$ outgoing from a, such that every $R_{m,n}$ represents a fragment of $\mathfrak{c}_0$ of length n and has a prefix of length $m+1$ representing fragments of consequent configurations of M, each having length $\leqslant 2n+3$. To complete the proof of the theorem we show that ontology $\mathcal{O}$ is expressible by a cyclic $\mathcal{EL}$-ontology network. Let us demonstrate that so is the family of axioms (4). Let $\varphi = A \sqsubseteq B$ be a concept inclusion and $B, B1, B_2$ concept names. By Lemma 12, ontology $\mathcal{O}_1 = \{\varphi[B \mapsto \exists r^k.B] \mid k \geqslant 0\}$ is expressible by a cyclic $\mathcal{EL}$-ontology network. Then by Lemma 2, ontology $\mathcal{O}_2 = \mathcal{O}_1[B \mapsto B_1 \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap B_2)]$ is expressible by a cyclic $\mathcal{EL}$-ontology network. By applying Lemma 12 again, we conclude that so is ontology $\mathcal{O}_3 = \bigcup_{l\geqslant 0} \mathcal{O}_2[B_1 \mapsto \exists v^l.B_1,\ B_2 \mapsto \exists(r,\mathtt{b})^{2l}.B_2]$, i.e., the ontology given by axioms $A \sqsubseteq \exists r^k.(\exists v^l.B_1 \sqcap \varepsilon \sqcap \exists r.(\mathtt{q_0} \sqcap \exists(r,\mathtt{b})^{2l}.B_2))$, for $k, l \geqslant 0$. Further, by Lemma 2, we obtain that $\mathcal{O}_2[B_1 \mapsto L,\ B_2 \mapsto \exists(r,\mathtt{b})^2]$ is expressible by a cyclic $\mathcal{EL}$-ontology network and hence, so is the family of axioms (4). A similar argument shows the expressibility of ontologies given by axioms (5)-(10). The remaining subset of axioms (11) of $\mathcal{O}$ is finite. By Lemma 1, there exists a cyclic $\mathcal{EL}$-ontology network $\mathcal{N}$ and an ontology $\mathcal{O}_{\mathcal{N}}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible and thus, it holds $\mathcal{O}_{\mathcal{N}} \models_{\mathcal{N}} A \sqsubseteq H$ iff M halts.

Theorem 3. *Entailment in $\mathcal{ALC}$-ontology networks is 2ExpTime-hard.*

Proof Sketch. We demonstrate that under a minor modification the construction from Theorem 7 in [7] shows that there exists a $\mathcal{ALC}$-ontology $\mathcal{O}$ and a concept name A such that $\mathcal{O} \not\models A \sqsubseteq \bot$ iff a given $\mathtt{1exp}(n)$-space bounded ATM M accepts the empty word. The ontology contains axioms with concepts of the form $\exists(r,C)^{\mathtt{1exp}(n)}.D$ and $\forall r^{\mathtt{1exp}(n)}.D$. By using Lemmas 4, 10, we show that every axiom of $\mathcal{O}$ containing concepts of size exponential in n is expressible by an acyclic $\mathcal{ALC}$-ontology network $\mathcal{N}$ of size polynomial in n. Then by applying Lemma 1 we obtain that there exists an acyclic $\mathcal{ALC}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_{\mathcal{N}}$ such that $\mathcal{O}$ is $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-expressible and thus, it holds $\mathcal{O}_{\mathcal{N}} \models_{\mathcal{N}} A \sqsubseteq \bot$ iff M accepts the empty word. Since AExpSpace = 2ExpTime, we obtain the required statement.

Theorem 4. *Entailment in $\mathcal{R}$-ontology networks is 3ExpTime-hard.*

Proof Sketch. The proof is by reduction of the word problem for $\mathtt{2exp}(n)$-space bounded ATMs to entailment in $\mathcal{R}$-ontology networks. Given such TM M and a number $n \geqslant 0$, we consider ontology $\mathcal{O}$ from the proof of Thm. 3 for M and let $\mathcal{O}'$ be the ontology obtained from $\mathcal{O}$ by replacing every concept of the form $\exists(r,C)^{\mathtt{1exp}(n)}.D$ and $\forall r^{\mathtt{1exp}(n)}.D$ with $\exists(r,C)^{\mathtt{2exp}(n)}.D$ and $\forall r^{\mathtt{2exp}(n)}.D$, respectively. A repetition of the proof of Thm. 3 using Lemmas 1, 8, 10 shows that there is an acyclic $\mathcal{R}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}'$ is $(\mathcal{N}, \mathcal{O}_\mathcal{N})$-expressible and thus, $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff M accepts the empty word. Since A2ExpSpace = 3ExpTime, we obtain the required statement.

Theorem 5. *Entailment in $\mathcal{ALCHOIF}$-ontology networks is coN2ExpTime-hard.*

Proof Sketch. We demonstrate that under a minor modification the construction from Theorem 5 in [7] shows that there exists a $\mathcal{ALCHOIF}$-ontology $\mathcal{O}$ and a concept name A such that $\mathcal{O} \not\models A \sqsubseteq \bot$ iff a given domino system admits a tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$, for $n \geqslant 0$. Ontology $\mathcal{O}$ contains axioms with concepts of the form $\exists r^{\mathtt{1exp}(n)}.C$ and $\forall r^{\mathtt{1exp}(n)}.C$. By using the same arguments as in the proof of Theorem 3 we show that there is a $\mathcal{ALCHOIF}$-ontology network $\mathcal{N}$ of size polynomial in n and an ontology $\mathcal{O}_\mathcal{N}$ such that $\mathcal{O}_\mathcal{N} \models_\mathcal{N} A \sqsubseteq \bot$ iff the domino system does not admit a tiling of size $\mathtt{2exp}(n) \times \mathtt{2exp}(n)$.

Theorem 6. *Entailment in $\mathcal{ROIF}$-ontology networks is coN3ExpTime-hard.*

Proof Sketch. The theorem is proved by a reducing the N3ExpTime-hard problem of domino tiling of size $\mathtt{3exp}(n) \times \mathtt{3exp}(n)$ to (non-)entailment in $\mathcal{ROIF}$-ontology networks. The proof employs a modification of the ontology $\mathcal{O}$ from the proof of Thm. 5 which is obtained like in the proof sketch to Thm. 4.

5 Membership Results

As a tool for proving upper complexity bounds, we demonstrate that entailment in a network $\mathcal{N}$ can be reduced to entailment from (a possibly infinite) union of 'copies' of ontologies appearing in $\mathcal{N}$. For an ontology network $\mathcal{N}$, let us denote $\mathtt{sig}(\mathcal{N}) = \bigcup_{\langle \mathcal{O}_1, \Sigma, \mathcal{O}_2 \rangle \in \mathcal{N}} (\mathtt{sig}(\mathcal{O}_1) \cup \Sigma \cup \mathtt{sig}(\mathcal{O}_2))$. An *import path* in $\mathcal{N}$ is a sequence $p = \{\mathcal{O}_0, \Sigma_1, \mathcal{O}_1, \dots, \mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n\}$, $n \geq 0$, such that $\langle \mathcal{O}_{i-1}, \Sigma_i, \mathcal{O}_i \rangle \in \mathcal{N}$ for each i with $(1 \leq i \leq n)$. We denote by $\mathtt{len}(p) = n$, $\mathtt{first}(p) = \mathcal{O}_0$ and $\mathtt{last}(p) = \mathcal{O}_n$ the *length* of p, the *first* and, respectively, the *last* ontologies on the path p. By $\mathtt{paths}(\mathcal{N})$ we define the set of all paths in $\mathcal{N}$, and by $\mathtt{paths}(\mathcal{N}, \mathcal{O}) = \{p \in \mathtt{paths}(\mathcal{N}) \mid \mathtt{first}(p) = \mathcal{O}\}$ the subset of paths that originate in $\mathcal{O}$. We say that $\mathcal{O}'$ is *reachable from $\mathcal{O}$ in $\mathcal{N}$* if there exists a path $p \in \mathtt{paths}(\mathcal{N}, \mathcal{O})$ such that $\mathtt{last}(p) = \mathcal{O}'$. The *import closure* of an ontology $\mathcal{O}$ in $\mathcal{N}$ is defined by $\bar{\mathcal{O}} = \cup_{p \in \mathtt{paths}(\mathcal{N}, \mathcal{O})} \mathtt{last}(p)$. Note that by definition it holds $\{\mathcal{O}\} \in \mathtt{paths}(\mathcal{N}, \mathcal{O})$ and thus, $\mathcal{O} \subseteq \bar{\mathcal{O}}$.

Lemma 13. *If $\mathcal{I} \models \bar{\mathcal{O}}$ then $\mathcal{I} \models_\mathcal{N} \mathcal{O}$.*

For every symbol $X \in \mathtt{sig}(\mathcal{N})$ and every import path p in $\mathcal{N}$, take a distinct symbol X_p of the same type (concept name, role name, or individual) not occurring in $\mathtt{sig}(\mathcal{N})$. For each import path p in $\mathcal{N}$, define a *renaming* θ_p of symbols in $\mathtt{sig}(\mathcal{N})$ inductively as follows. If $\mathtt{len}(p) = 0$, we set $\theta_p(X) = X$ for every $X \in \mathtt{sig}(\mathcal{N})$. Otherwise, $p = p' \cup \{\mathcal{O}_{n-1}, \Sigma_n, \mathcal{O}_n\}$ for some path p' and we define $\theta_p(X) = \theta_{p'}(X)$ if $X \in \Sigma_n$ and $\theta_p(X) = X_p$ otherwise. A *renamed import closure* of an ontology $\mathcal{O}$ in $\mathcal{N}$ is defined by $\tilde{\mathcal{O}} = \bigcup_{p \in \mathtt{paths}(\mathcal{N}, \mathcal{O})} \theta_p(\mathtt{last}(p))$.

Lemma 14. *If $\mathcal{I} \models \tilde{\mathcal{O}}$ then $\mathcal{I} \models_\mathcal{N} \mathcal{O}$.*

Lemma 15. *For every $\mathcal{I} \models_\mathcal{N} \mathcal{O}$ there exists $\mathcal{J} \models \tilde{\mathcal{O}}$ such that $\mathcal{J} =_{\mathtt{sig}(\mathcal{N})} \mathcal{I}$.*

Theorem 7. *Let $\mathcal{N}$ be an ontology network, $\mathcal{O}$ an ontology in $\mathcal{N}$, and α an axiom such that* $\texttt{sig}(\alpha) \subseteq \texttt{sig}(\mathcal{N})$. *Then $\mathcal{O} \models_{\mathcal{N}} \alpha$ iff $\tilde{\mathcal{O}} \models \alpha$.*

Theorem 7 provides a method for reducing the entailment problem in ontology networks to entailment from ontologies. Note that, in general, the renamed closure $\tilde{\mathcal{O}}$ of an ontology $\mathcal{O}$ in a (cyclic) network $\mathcal{N}$ can be infinite (even if $\mathcal{N}$ and all ontologies in $\mathcal{N}$ are finite). There are, however, special cases when $\tilde{\mathcal{O}}$ is finite. For example, if all import signatures in $\mathcal{N}$ include all symbols in $\texttt{sig}(\mathcal{N})$, then it is easy to see that $\tilde{\mathcal{O}} = \bar{\mathcal{O}}$. $\tilde{\mathcal{O}}$ is also finite if $\texttt{paths}(\mathcal{N}, \mathcal{O})$ is finite, e.g., if $\mathcal{N}$ is acyclic. In this case, the size of $\tilde{\mathcal{O}}$ is at most exponential in $\mathcal{O}$. If there is at most one import path between every pair of ontologies (i.e., if $\mathcal{N}$ is tree-shaped) then the size of $\tilde{\mathcal{O}}$ is the same as the size of $\mathcal{N}$. This immediately gives the upper complexity bounds on deciding entailment in acyclic networks.

Theorem 8. *Let $\mathcal{L}$ be a DL with the complexity of entailment in [co][N]TIME$(f(n))$ ([co] and [N] denote possible co- and N-prefix, respectively). Let $\mathcal{N}$ be an acyclic ontology network and $\mathcal{O}$ an ontology in $\mathcal{N}$ such that $\tilde{\mathcal{O}}$ is a $\mathcal{L}$-ontology. Then for $\mathcal{L}$-axioms α, the entailment $\mathcal{O} \models_{\mathcal{N}} \alpha$ is decidable in [co][N]TIME$(f(2^n))$. If $\mathcal{N}$ is tree-shaped then deciding $\mathcal{O} \models_{\mathcal{N}} \alpha$ has the same complexity as entailment in $\mathcal{L}$.*

The next theorem is proved by the compactness theorem for First-Order Logic by showing that the renamed import closure of an ontology is recursively enumerable.

Theorem 9. *Let $\mathcal{L}$ be a DL, which can be translated to FOL, $\mathcal{N}$ an ontology network, and $\mathcal{O}$ an ontology in $\mathcal{N}$ such that $\tilde{\mathcal{O}}$ is a $\mathcal{L}$-ontology. Then for $\mathcal{L}$-axioms α, the entailment $\mathcal{O} \models_{\mathcal{N}} \alpha$ is semi-decidable.*

6 Conclusions

We have introduced a new mechanism for ontology integration based on semantic import relations between ontologies. Reasoning over ontologies with semantic imports can be reduced to reasoning over the union of ontologies obtained as 'copies' of ontologies from the import closure under an injective renaming of signature symbols. We have shown that this gives an upper bound on the complexity of reasoning with acyclic semantic imports, which is one exponential harder than entailment in the underlying DL, from $\mathcal{EL}$ to $\mathcal{SROIQ}$. Our hardness results demonstrate that the increased complexity of reasoning is unavoidable. When cyclic imports are allowed, the complexity jumps to undecidability, even if every ontology in a combination is given in the DL $\mathcal{EL}$. These complexity results are shown for situations when the imported symbols include roles. It is natural to ask whether the complexity drops when only concept names are imported. The second parameter which may influence the complexity of reasoning is the semantics which is 'imported'. In the proposed mechanism, importing the semantics of symbols is implemented via agreement of models of ontologies. One can consider refinements of this mechanism, e.g., by carefully selecting the classes of models of ontologies which must be agreed. The third way to decrease the complexity is to restrict the language in which ontologies are formulated. We conjecture that reasoning with cyclic imports is decidable for ontologies formulated in the family of DL-Lite.

References

1. Bao, J., Voutsadakis, G., Slutzki, G., Honavar, V.: Package-based description logics. In: Modular Ontologies, pp. 349–371 (2009)
2. Borgida, A., Serafini, L.: Distributed description logics: Assimilating information from peer sources. J. Data Semantics pp. 153–184 (2003)
3. Euzenat, J., Zimmermann, A., de Freitas, F.L.G.: Alignment-based modules for encapsulating ontologies. In: Proceedings of the 2nd International Workshop on Modular Ontologies, WoMO 2007, Whistler, Canada, October 28, 2007 (2007), http://ceur-ws.org/Vol-315/paper4.pdf
4. Grau, B.C., Motik, B.: Reasoning over ontologies with hidden content: The import-by-query approach. J. Artif. Intell. Res. (JAIR) pp. 197–255 (2012)
5. Grau, B.C., Parsia, B., Sirin, E.: Ontology integration using epsilon-connections. In: Modular Ontologies, pp. 293–320 (2009)
6. Homola, M., Serafini, L.: Towards formal comparison of ontology linking, mapping and importing. In: Description Logics (2010)
7. Kazakov, Y.: $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are harder than $\mathcal{SHOIQ}$. In: Proc. 11th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'08). pp. 274–284. AAAI Press (2008)
8. Pan, J.Z., Serafini, L., Zhao, Y.: Semantic import: An approach for partial ontology reuse. In: Haase, P., Honavar, V., Kutz, O., Sure, Y., Tamilin, A. (eds.) WoMO. CEUR Workshop Proceedings, vol. 232. CEUR-WS.org (2006), http://dblp.uni-trier.de/db/conf/semweb/womo2006.html