Software Engineering, Theory and Experimental Programming

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Expressibility of Languages with Capture Operations: an Overview

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A Precursor: Pattern Languages

Dana Angluin, 1980: Finding Patterns Common to a Set of Strings.

- Letters in an alphabet Σ;
- Variables: matching against strings in Σ^*.

Non-trivial language inclusion for finite alphabets (undecidable for erasing case): $XabY$ and $XaZbY$ match against the same set of strings in $\{a, b\}^*$, i.e. $\mathscr{L}(XabY) = \mathscr{L}(XaZbY)$ if $\Sigma = \{a, b\}$.

Angluin's infamous theorem

Matching a string against a pattern is NP-complete.

In practice:

- Efficient exact matching techniques for restricted classes of patterns (regular, bounded-width).
- Techiques for approximate matching.

Extended Regular Expressions

Evolved from 1980s and Perl regular engine:

- lookaheads, lookbehinds — positive, negative. Positive lookahead syntax:
  $(? = r_1)r_2$ (matching a regex r_2 whose prefix satisfies r_1. For readability, sometimes we assume that lookahead matches against the whole string)

- capture groups and backreferences (recursive and not). Syntax:
  $(aa^+)\backslash 1^+$ — capture groups are numbered by parentheses ordering. Lookaheads are not capture groups.

- recursive definitions. Recursive definition syntax:
  $(a(?1)b|c)$ — the syntax (not the matched string) of 1-st capture group is reused.

Regex Classes Currently Investigated

- Non-recursive backref-regexes ($\mathcal{RE}w\mathcal{BR}^-$): Campeanu–Salomaa–Yu formalism (2003);
- Recursive backref-regexes ($\mathcal{RE}w\mathcal{BR}$): Shmid formalism, memory finite automata (2016);
- Recursive backref-regexes with lookahead $\mathcal{RE}w\mathcal{BR} + \mathcal{LA}$: Chida–Terauchi extended MFA formalism (2023).

Backreferences Formalisations

- Appeared much later than implementations of backref-regexes.
- Some almost repeated PCRE (Perl) backref-regexes.

Campeanu–Salomaa-Yu (CSY) formalisation:

$$\begin{cases} (\tau) & \text{(anonymous capturing)} \\ \backslash k & \text{(reading memory cell from k-th group)} \end{cases}$$

Example: $(\texttt{aa}^+)(\backslash\texttt{1})^+$ defines $\{a^n \mid \text{n is not prime}\}$

- Any capture group is initialized exactly once.
- Any reference must be preceded by the capture group textually.

Backreferences Formalisations

- Currently most used in the theoretical scope: only named capture groups.

Backref-regex (ref-words, by Shmid) operations:

$$\begin{cases} [_k \tau]_k & \text{(named capturing)} \\ \&k & \text{(reading memory cell)} \end{cases}$$

Example: $[_1\texttt{a}^*]_1\texttt{a}^+\texttt{b}\&\texttt{1}$ defines $\{a^m b a^n \mid m > n\}$

- ε-semantics (Schmid) — uninitialized reference recognizes $\{\varepsilon\}$;
- $\varnothing$-semantics (regex engines) — uninitialized reference recognizes $\varnothing$.

No impact on language properties.

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Example: $[_1\mathtt{a}^*]_1\mathtt{a}^+\mathtt{b}\&\mathtt{1}$ defines $\{a^m b a^n \mid m > n\}$

- Possibly unbalanced and nested (but not self-nested) capturing.
- References on k-th memory cell cannot occur inside a capturing group for k.

Word Equations

- Given a pair of patterns Φ, Ψ sharing common variables $X_1, \ldots, X_n$, we say a tuple $\langle \omega_1, \ldots, \omega_n \rangle$ is a solution set to equation $\Psi \doteq \Phi$ if a morphism defined by $X_i \mapsto \omega_i, \gamma \mapsto \gamma$ turns both patterns into equal strings.

- Given equation $E : aX_1X_1bX_2 \doteq X_1aX_2bX_1$, the set $\{\langle a^n, (a^nb)^m a^n \rangle \mid m, n \in \mathbb{N}\}$ is the solution set to E.

  Note: X_2-projection of the set is not context-free. Still, the X_2-projection belongs to the class of Okhotin's conjunctive languages (discussed later).

Word Equations

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- A k-projection of a solution set $\big\{ \langle \omega_1, \ldots, \omega_n \rangle \mid \Phi[\omega_1, \ldots, \omega_n] = \Psi[\omega_1, \ldots, \omega_n] \big\}$ is the set of k-components of the tuples.
- If a string set $\mathscr{L}$ is a projection of the solution set of a word equation $\Phi \doteq \Psi$, then $\mathscr{L}$ is said to be a language of $\Phi \doteq \Psi$.

Captured Values and Patterns

A word equation over k variables defines k languages.
As a pattern (i.e. a pair of patterns defining the same string), a word equation $\Phi \doteq \Psi$ defines a single language, as usual, $\{\Phi[\omega_1, \ldots, \omega_n] \mid \Phi[\omega_1, \ldots, \omega_n] = \Psi[\omega_1, \ldots, \omega_n]\}$.

We distinguish the first languages class $\mathcal{WE}_\pi$ and the second $\mathcal{WE}$ and note that $\mathcal{WE} \subset \mathcal{WE}_\pi$, since $a^* \in \mathcal{WE}_\pi$, but every equation defining such a variable value must have at least one explicit a-occurrence, hence its side after any substitution cannot be valued ε.

$\mathcal{REwBR}$ and $\mathcal{REwBR}_\pi$

$\mathcal{REwBR} \subset \mathcal{REwBR}_\pi$

Given $\mathscr{L} = \{a^n b^n \mid n \in \mathbb{N}\}$, it can be constructed as a captured value $\&3$ in the ref-word

$$\big([_2 a\&1 b]_2 [_1 a\&2 b]_1\big)^* [_3 \& \mid \&2]_3$$

$\mathscr{L}$ is not expressible by a $\mathcal{REwBR}$: both a-block and b-block have to contain an iteration $\Rightarrow$ asynchronous growth.

Known Analysis Techniques

- $\mathcal{REwBR}$ and $\mathcal{REwBR} + \mathcal{LA}$, known proofs refer to the restricted CSY formalisation $\Rightarrow$ no known formal techniques of disproving expressibility for multiple capturing case.

- For $\mathcal{WE}$, W. Plandowski in 2000s developed a combinatorial method relying on synchronising $\mathcal{F}$-codes.

- In 2023, J. Day used the method to prove that all the star subexpressions Φ^* of a thin regular language recognised by a word equation language has the following property: every two elements ω_1, ω_2 of the language recognised by Φ commute, i.e. $\omega_1\omega_2 = \omega_2\omega_1$.

$\mathcal{REwBR}$ and $\mathcal{REwBR} + \mathcal{LA}$: Growth Argument

The factorial star language $\{(a^{k!})^+ba^k\}$:

$$\left((?{=}\ [_2a\&1]_2a^*b)(?=[_1\&2]_1a^*b)(?{=}\ \&2^*b)\right)^* a^*b\&2$$

Main reason of non-expressibility in $\mathcal{REwBR}$: given a maximal length multiplier N^N (i.e. given length of the regex N), the words $(a^{N^N!})^+ba^{N^N}$ cannot be expressed, because of the abnormal growth of the first a-block.

In Chida's paper, the CSY formalism was referred with a non-expressible language $\left\{a^iba^{i+1}ba^{i\cdot(i+1)\cdot k} \mid i, k \in \mathbb{N}\right\}$.

REGEX and $\mathcal{CNJ}$

REGEX includes conjunctive languages by Okhotin expressible by the rules augmented with conjunction:

$$\{A_i \to \Phi_1 \,\&\, \cdots \,\&\, \Phi_2 \mid A_i \in N \,\&\, \Phi_j \in (N \cup \Sigma)^*\}$$

Example: a grammar for $\{(a^n b)^k \mid n, k \in \mathbb{N}\}$.

$$\begin{aligned} S &\to SA \,\&\, Cb \mid A \\ A &\to aA \mid ab \\ C &\to aCa \mid B \\ B &\to BA \mid b \end{aligned}$$

May be expressed with the use of recursion and lookaheads:

$$\Big(\overbrace{(?= \underbrace{(a(?2)a|b|b.^*b)}_{\text{capture group 2}} b) a^* b (?1) | a^* | \underbrace{(a(?3)a|b)}_{\text{group 3}}}^{\text{capture group 1}}\Big) b$$

Technique: same to μ-regexes for CFG, augmented with lookaheads.

Linear $\mathcal{CFL}$ and $\mathcal{REwBR}_\pi$

A language is linear-context-free $\Leftrightarrow$ it is expressed by a CFG having at most one non-terminal in each rule rhs.

Every linear $\mathcal{CFL}$ can be expressed by a capture group language.

- For every non-terminal N_i, introduce a pair of memory cells i, i' (for capturing in an lhs and for recapturing).
- Each recursive rule $N_i \to \omega_1 N_j \omega_2$ is rewritten into a subexpression $r_k : [_{i'}\omega_1 \& j\omega_2]_{i'} [_i \& i']_i$. Terminal rules for every nonterminal N_i are gathered into a disjunction and are rewritten into an expression r'_i in the similar way.
- A total expression is $r'_1 \dots r'_m (r_1 | \dots | r_k)^*$.

$\mathcal{CFL}$ and $\mathcal{REwBR}_\pi$

Some non-linear (and non-linear-conjunctive) languages such as $\{(ww^R)(vv^R) \mid w, v \in \Sigma^*\}$ can be generated by cells of $\mathcal{REwBR}$. Some non-linear context-free languages cause encoding problems (e.g. balanced parentheses language). Reason: unbounded treewidth of the derivation tree.

On the other hand, using the $\mathcal{CNJ} \subset$ REGEX technique, the balanced parenthesis language can be expressed by $\mathcal{REwBR}+\mathcal{LA}_\pi$. Idea: construct a Trellis automaton and transform it to a linear $\mathcal{CNJ}$.

d	a	b	d	r
a	a	d	a	a
b	r	b		r
d		b		a
r	r	b	b	r

$\mathcal{WE}_\pi$ and $\mathcal{REwBR} + \mathcal{LA}_\pi$

- Given $\Phi(X_1, \ldots, X_n) \doteq \Psi(X_1, \ldots, X_n)$ with a canonical numeration of variables, every first occurrence of a variable in $\Phi\Psi$ is replaced with a capture group $(.^*)$, while the other variable occurrences are replaced by $\backslash k$ operations. The resulting equation sides are Φ', Ψ'.
- Regex $(?{=}\ \Phi')\Psi'$ recognises the solution projections in its capture groups.

Main reason: capture-preserving lookaheads in REGEX.

For instance, in the expression $\big(?{=}\ (.^*)ab(.^*)\big)\backslash 2ba\backslash 1$ both first and second memory cells are captured by the lookahead checker.

$q\mathcal{WE}_\pi$ and $\mathcal{RE}w\mathcal{BR}_\pi$

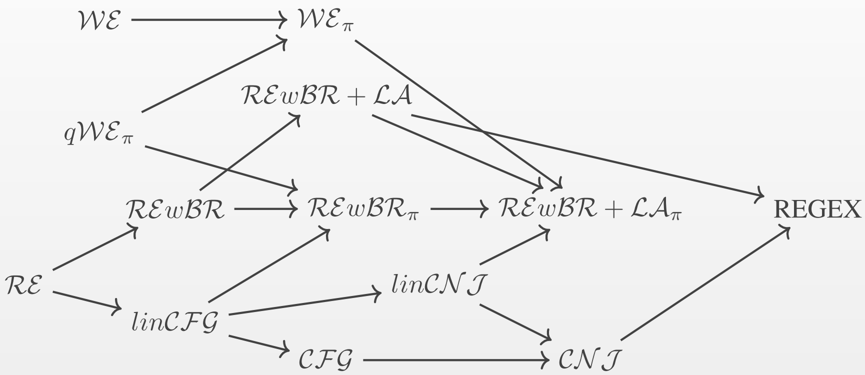
In a quadratic word equation, each variable occurs at most twice.

- Every solution to a quadratic word equation is expressed by a graph containing the assignments: $X \mapsto YX$, $X \mapsto Y$, $X \mapsto aX$, $X \mapsto \varepsilon$.
- Can be transformed to an MFA moving from graph leaves to its root with re-capturing. E.g. $X \mapsto YX$ is represented by $[_{X'} \& Y \& X]_{X'} [_X \& X']_X$ subexpression.

Variable values in $XabY \doteq YbaX$ are captured by

$$([_X a^*]_X [_Y \& X a]_Y | [_Y b^*]_Y [_X \& Y b]_X) \left([_{X'} \& Y ba \& X]_{X'} [_X \& X']_X | [_{Y'} \& X ab \& Y]_{Y'} [_Y \& Y']_Y \right)^*$$

Summary: What is Known?



Open Problems

- Techniques for disproving membership in $\mathcal{RE}w\mathcal{BR}$, $\mathcal{RE}w\mathcal{BR} + \mathcal{LA}$ classes?
- Clarifying position of word-equation-languages in the regex languages hierarchy: $\mathcal{RE}w\mathcal{BR}$, $\mathcal{RE}w\mathcal{BR}_\pi$, $\mathcal{RE}w\mathcal{BR} + \mathcal{LA}$ or $\mathcal{RE}w\mathcal{BR} + \mathcal{LA}_\pi$?
- Determining a cone over $\mathcal{WE}$ and $\mathcal{RE}w\mathcal{BR}$: a minimal language class closed under intersections with regular languages, homomorphism and inverse homomorphism operations.

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