Calculating Fibonacci numbers using the Binet formula without using floating point arithmetic

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By now, many ways have been invented for calculating Fibonacci numbers: from direct recursion based on the formula:

$$F_n=F_{n-1}+F_{n-2}$$

to the matrix method described by D. Knuth

Binet's formula is located separately in this series of algorithms, which has the form:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

This formula seems attractive at first glance, but it contains an irrational number, which in computer calculations we are forced to represent in the form of a floating point number (i.e. replace an infinite non-periodic fraction with a finite one).

This means that the calculations will not be accurate; a limitation error is introduced into them.

The author once came across a publication in which Binet's formula was used to calculate a very large Fibonacci number, but the implementation assumed the use of superhigh-bit floating arithmetic (so that the required number would completely fit into the mantissa).

We'll take a completely different path!

Consider a set of numbers of the form:

$$x = a + \sqrt{5} * b$$

where a and b are integers. It seems quite obvious that this set is algebraically closed with respect to the operations of ordinary addition and multiplication:

$$\left(a + b\sqrt{5}\right) + \left(c + d\sqrt{5}\right) = \left((a + c) + (b + d)\sqrt{5}\right)$$

$$\left(a + b\sqrt{5}\right)\left(c + d\sqrt{5}\right) = \left((ac + 5bd) + (ad + bc)\sqrt{5}\right)$$

In addition, zero and unit belong to the set under consideration in a trivial way:

$$1 \equiv (1 + \sqrt{5} * 0)$$

$$0 \equiv (0 + \sqrt{5} * 0)$$

Subtraction is quite naturally realized:

$$\left(a + b\sqrt{5}\right) - \left(c + d\sqrt{5}\right) = \left((a - c) + (b - d)\sqrt{5}\right)$$

Now you can implement arithmetic on a set of pairs (a, b), in which addition, subtraction and multiplication will be described by the formulas:

$$(a,b)+(c,d)=((a+c),(b+d))$$

$$(a,b)-(c,d)=((a-c),(b-d))$$

$$(a,b)*(c,d)=((ac+5bd),(ad+bc))$$

Thus, we can "safely forget" about √5 and implement a direct calculation using the Binet formula.

As a result, the numerator of the fraction will be a pair of the form (0, r√5) = r√5. Dividing this irrational number by √5 gives the desired integer result. Naturally, in reality, dividing is not required, it is enough to calculate (using the above-described pair arithmetic) two binomials:

$$A = \frac{(1+\sqrt{5})^n}{2^n} \quad и \quad B = \frac{(1-\sqrt{5})^n}{2^n}$$

and then subtract A-B

```python
def prod_pairs(a,b): # pairs multiplication
    return (a[0]*b[0]+5*a[1]*b[1],a[0]*b[1]+a[1]*b[0])
def sub_pairs(a,b):   # pairs subtracting
    return (a[0]-b[0],a[1]-b[1])
def pow_pair(a,n):   # exponentiation
    c=a
    for _ in range(n-1):
        c=prod_pairs(c,a)
    return c
def fib_bine(n): # Binet formula
    x1=pow_pair((1,1),n)
    x2=pow_pair((1,-1),n)
    z=sub_pairs(x1,x2)
    return z[1]//(2**n)
```

Is it possible to speed up this code? Yes, we can, if we speed up the exponentiation.

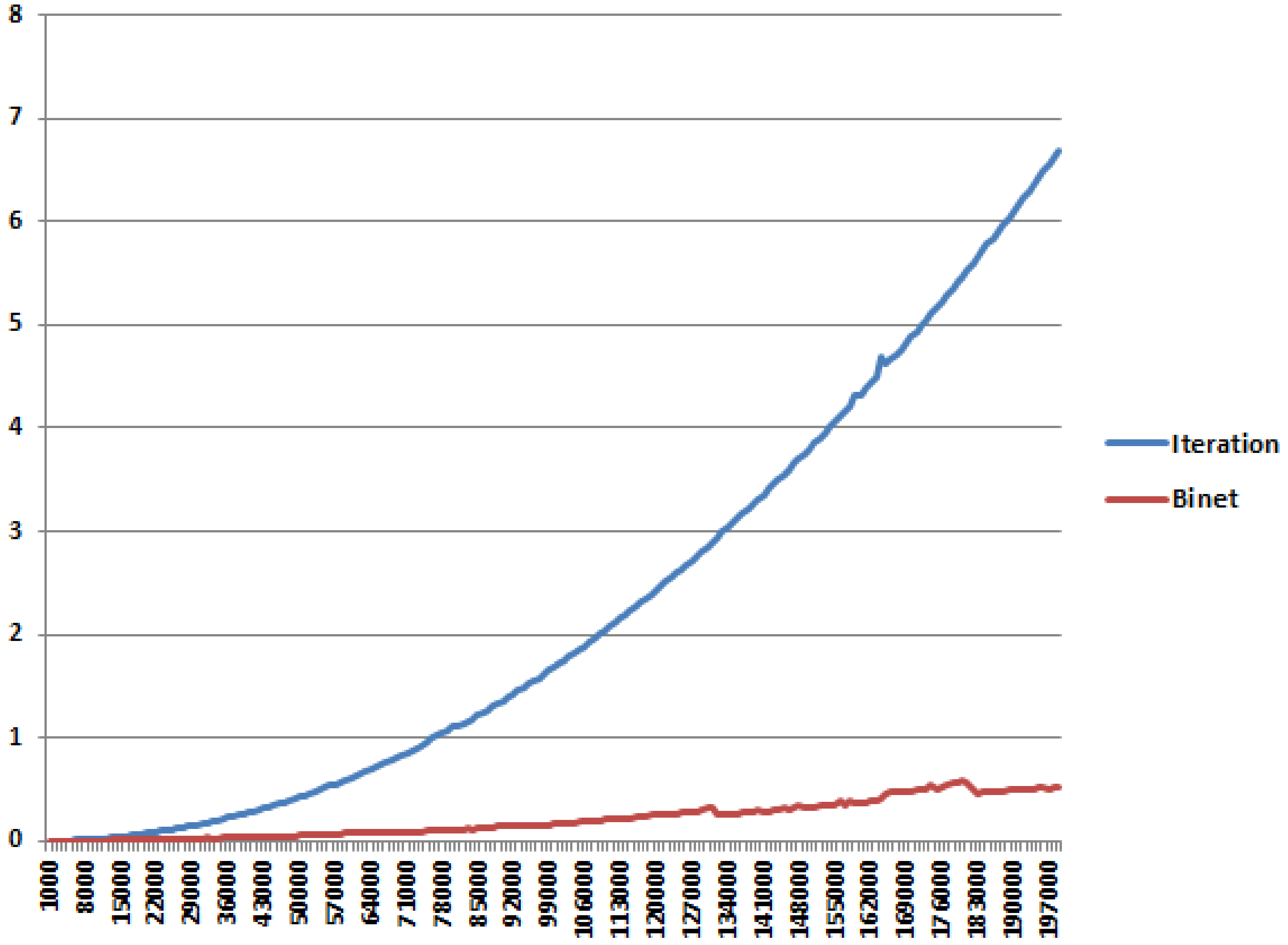
To speed up exponentiation, there is a standard approach, which is that to calculate x^n, the chain $x \to x^2 \to x^4 \to \ldots \to x^{2^k}$ is calculated until $2^k \le n$, and then $x^{(n - 2^k)}$.

```
def pow_pair(a,n):
    if (n==1):
        return a
    c=copy(a)
    k=1
    while k*2<=n:
        if k<=n:
            c=prod_pairs(c,c)
            k=k*2
    p=n-k
    if p>=1:
        tmp=pow_pair(a,p)
        return prod_pairs(tmp,c)
    else:
        return c
```

Using this technique allows us to calculate Fibonacci numbers in a logarithmic time using the Binet formula and without using floating point arithmetic.

Below are the test results comparing the computation time of Fibonacci numbers by simple iteration:

```
def fib_ite(n):
    c,p=0,1
    for _ in range(n):
        c,p=c+p,c
    return c
```


8
7
6
5
4
3
2
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0
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134000
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148000
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176000
183000
190000
197000
Iteration
Binet

Despite the apparent simplicity of the fib_ite code, the fib_bine function performs significantly better.

Thanks for your attention!