Finally, what is the complexity of Dijkstra's assignment algorithm?

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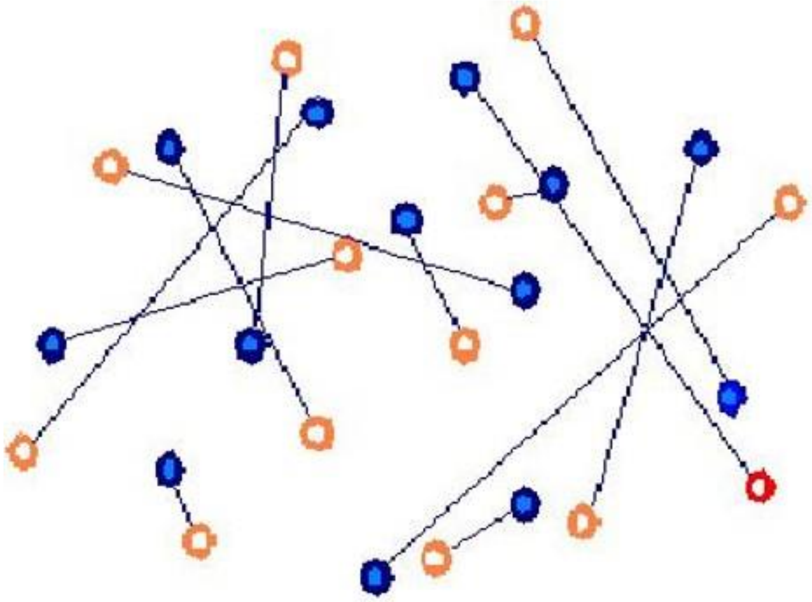
Summer School Marktoberdorf

(https://en.wikipedia.org/wiki/Summer_School_Marktoberdorf)

- The International Summer School Marktoberdorf is an annual two-week summer school for international computer science and mathematics postgraduate students and other young researchers, held annually since 1970 in Marktoberdorf, near Munich in southern Germany. The summer school is supported as an Advanced Study Institute of the NATO Science for Peace and Security Program. It is administered by the Faculty of Informatics at the Technical University of Munich.

Dijkstra's assignment problem

- There are $n > 0$ black and $n > 0$ white points on the plane without collinear triples. Proof that it is possible to couple black and white pairwise by segments without intersections.



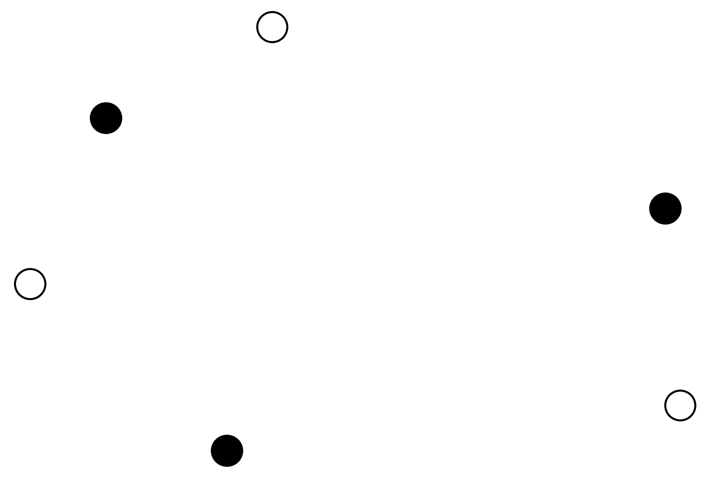
How to solve it?

- MRP is also closely related to (and may be solved using)
  - the Assignment Problem in Graph Theory
  - to the Convex Hull Problem in Combinatorial Geometry
- “Heuristic” but formally verified local search solution was suggested by E.W. Dijkstra at Summer School Marktoberdorf 1994:

Dijkstra’ approach

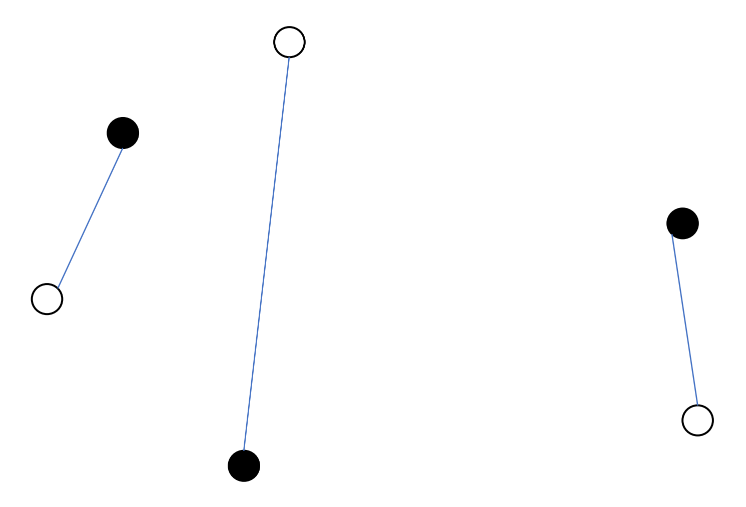
- There are n black and n white points on the plain “in general position” (without collinear triples).
- Design an algorithm to couple the input points by intersection-free segments.

Example of input data:

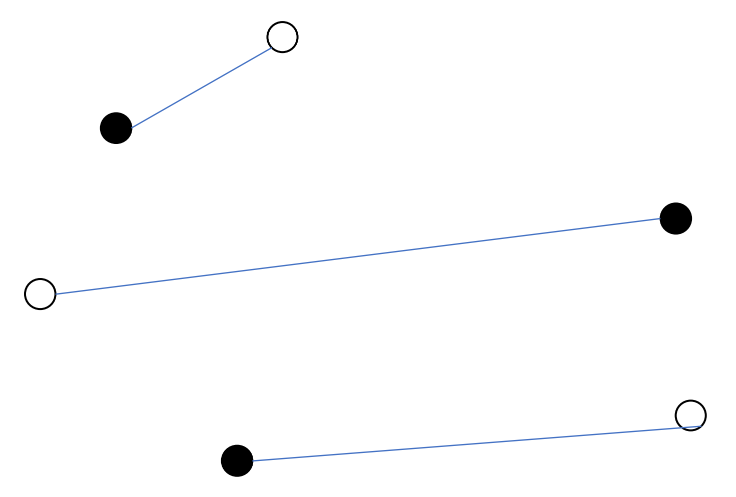


Dijkstra’ approach (cont.)

Example-1 of output data:



Example-2 of output data:



Dijkstra’ algorithm

```
/*ASSUME: points are in general position.*/
VAR X: coupling;
X:=FIRST;
WHILE not(FREE(X)) DO X:=SWAP(X)
/*Assert: X is intersection-free*/
```

Data type COUPLING

- Coupling is a data type that values are collections of segments for given points,
- function FIRST: $\rightarrow$ coupling returns a value of this type,
- function FREE: coupling $\rightarrow$ Boolean checks intersections,
- function SWAP: coupling $\rightarrow$coupling resolves a local conflict (if any).

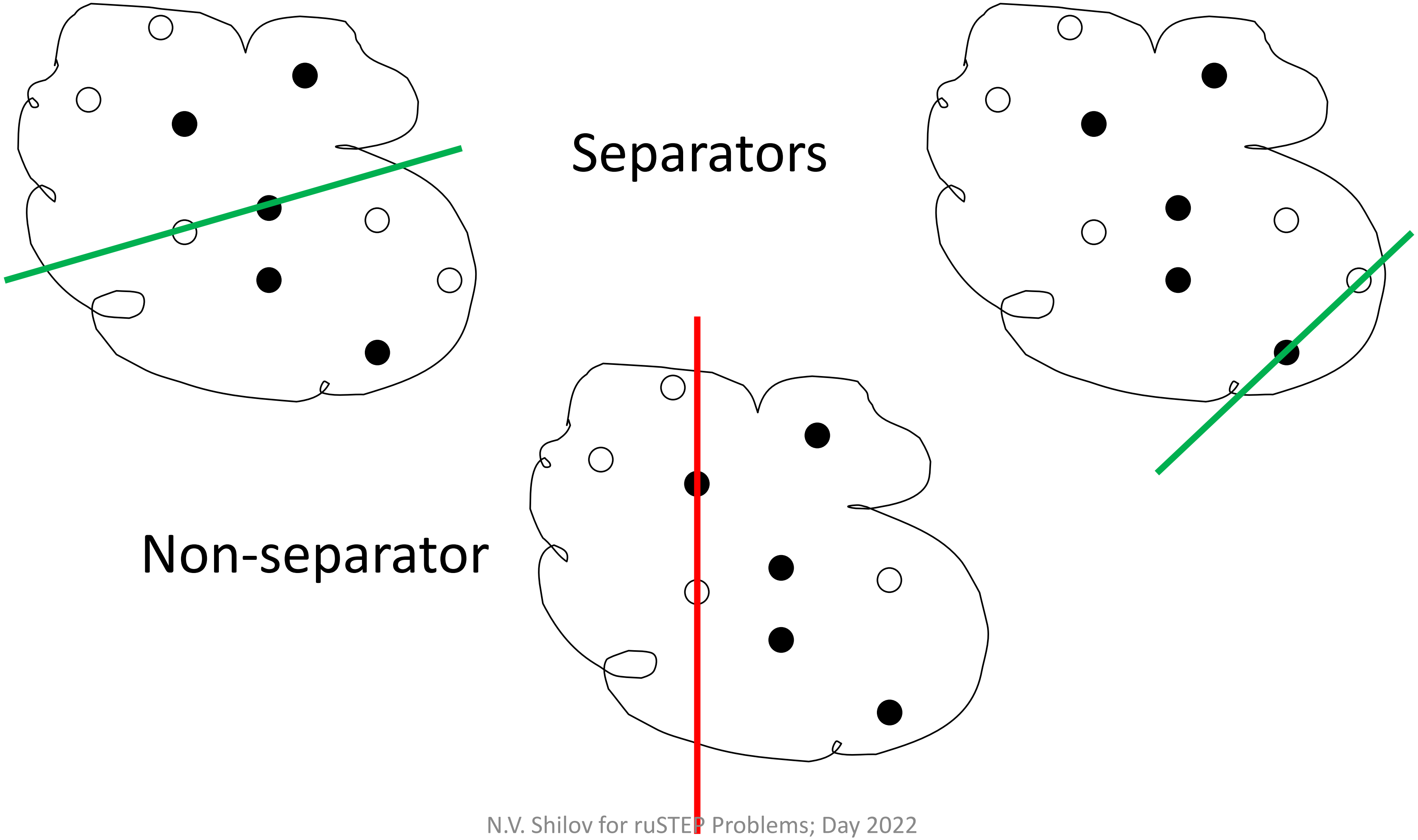
Idea how to prove Dijkstra algorithm termination

- Assign real value $L(X)= \Sigma_{[b,w]\in X}|[a,b]|$ to each coupling X,
- remark that the set of couplings (for given input points) is finite,
- and that each legal SWAP reduce L(X).

Alternative idea how to solve Dijkstra assignment problem

- There must be a *separator* line that
  - comes through a black and a white points,
  - divide the plain onto “halves”, i.e., “left” and “right” semi-plains,
  - numbers of black and white points in the left semi-plain are equal,
  - and numbers of black and white points in the right semi-plain are equal.
- Exercise: Prove that every set of n black and n white points in general position has a separator line.

Alternative idea how to solve Dijkstra assignment problem (cont.)



Algorithmic (recursive) implementation of the alternative idea

GOOD(Black, White) = CASE OF:

$|Black| \neq |White|$ THEN return $\varnothing$.

Black$\cup$White aren't in general position THEN return $\varnothing$.

$|Black| = |White| = 0$ THEN return $\varnothing$.

OTHERWISE

let B$\in$Black and W$\in$White be points that

line (B,W) is a separator for Black$\cup$White

in return {[B,W]} $\cup$ GOOD(Black right to (B,W), White right to (B,W)) $\cup$

$\cup$ GOOD(Black left to (B,W), White left to (B,W)).

Performance analysis of the alternative solution

- Let us consider case when arguments of the function FREE meet the following properties:
  - |Black|=|White| and
  - Black$\cup$White are in general position.
- If to adopt |Black| as the data complexity measure, then the werst-case time complexity satisfies the following inequality:

$$T(n) \leq \text{if } n = 0 \text{ then } O(1) \text{ else } O(1) \; + $$
$$+ \text{ time to find a separator for } n \text{ black and } n \text{ white points} +$$
$$+ \max_{0 \leq k < n} \left(T(k) \; + \; T(n-1-k) \right).$$

Performance analysis of the alternative solution (cont.)

- Hence for all $n \in \boldsymbol{N}$ we have

  $$T(n) \leq \text{if } n = 0 \text{ then } O(1) \text{ else } O(n^3) + \max_{0 \leq k < n} (T(k) + T(n-1-k)).$$

- Since $T(n) \leq \text{if } n < const \text{ then } O(1)$

  $$\text{else } O(n^m) + \max_{0 \leq k < n} (T(k) + T(n-1-k))$$

  implies that $T(n) = O(n^{m+1})$, the alternative solution has time complexity $O(n^4)$.

Related Problems: SMP – Stable Marriage Problem

- Given n men and n women, where each person has a wish list, i.e., it ranks all persons of the opposite sex with a unique number in $[1..n]$.
- Problem: Marry the men and women together such that there are no two people of opposite sex who would both rather have each other than their current partners. (If there are no such people, all the marriages are *stable*.)

Related Problems: SMP – Stable Marriage Problem (cont.)

- In 1962, David Gale and Lloyd Shapley proved that it is always possible to solve the SMP by a centralized (i.e., not a multiagent) algorithm where men and women are rather dolls than agents.

```
INITIALISE all M and W to be free ;
   WHILE there is any free M who still has a free W in his
list DO
LET M be any free man
IN LET W be the woman in his list with highest rank
whom he has not proposed engagement yet
      IN IF W is free
            THEN make M and W engaged
            ELSE LET M’ be a man engaged with W
            IN IF W prefers M to M’
                  THEN make M and W engaged and free M’
                  ELSE M’ and W remain engaged
```

Related Problems: MRP – Mars Robot Puzzle

- There are $n > 1$ autonomous agents (*robots*) and (the same) number of *shelters* on a plane part of Mars.
- Locations of all shelters are fixed and known to all robots.
- Every robot could communicate with any other robot in P2P manner.
- Every robot knows its own actual position but is not aware about positions of other robots.

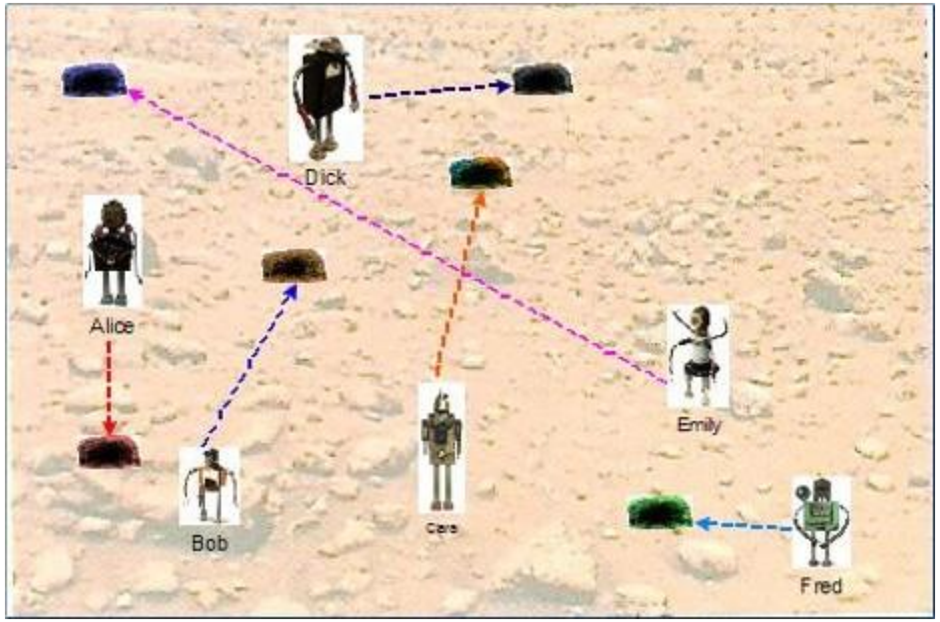
Планета Шелезяка. Полезных ископаемых — нет, воды — нет, растительности — нет. Населена роботамиShelezyak's planet. There are no minerals, no water, no vegetation. Inhabited by robots. (Тайна третьей планеты, https://citaty.info/quote/347595)

Related Problems:
MRP – Mars Robot Puzzle (cont.)

- At some moment, all robots stop (fix their current positions), and must select individual shelters to move at by a *straight route*.
- Assume that there are no any obstacle (like rocks, holes, robots and shelters, etc.) between any robot and any shelter.
- Definitely, robots should not collide (it means that their routes should not intersect).
- Hence, every individual robot can move to its shelter only when it knows for sure that it will not collide with any other robot on the route.

Related Problems: MRP – Mars Robot Puzzle (cont.)

- Problem: Design a multiagent algorithm that guarantees that every robot will eventually know that its route to the selected shelter does not intersect with routes of other robots (and hence robots will not collide in a motion).

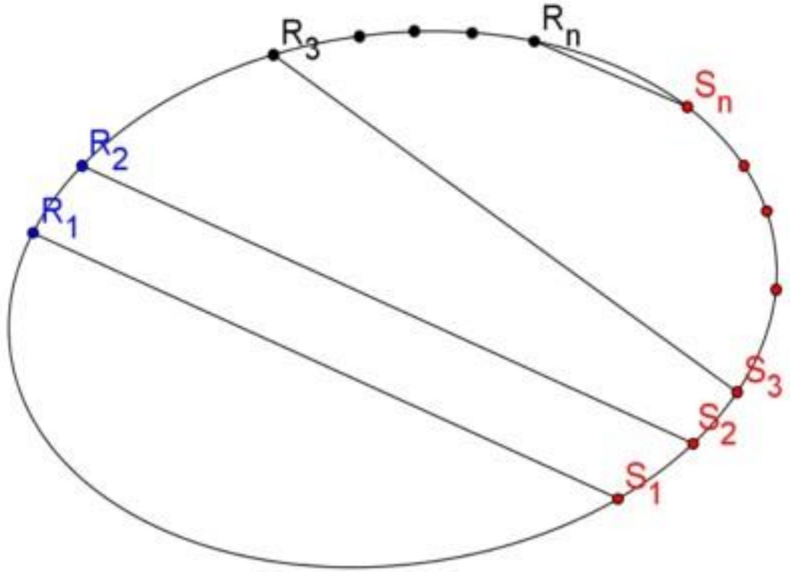


Related Problems: MRP – beliefs and knowledge of the Agents

- Individual *beliefs* are represented by two counters:
  - NC (Number of Conflicts) represents an upper estimation of number of partners with whom it could have conflicts – an agent believes that it has no races when NC$= 0$.
  - CF (Conflict Free) represents a lower estimation of number of agents that have no conflicts at all – it believes that there is no any race when NC$= 0$ and CF $= (n - 1)$.
- An agent *knows* that there is no any race when NC$= 0$ and CF $= 2(n - 1)$ (i.e., it has no rivals, and it checks twice that all other agents believe that they do not compete with anyone.)

Related Problems: MRP – information complexity (FYI)

- Assume that $L \subseteq \boldsymbol{R}^2$ is a continuous closed line that bounds a convex region, π is a protocol that solves MRP for n robots and n shelters in L, and $m \in \boldsymbol{N}$ is a constant.
- Then m can't be an upper bound for the number of bits that robots send each other executing π.



Hypothesis:
Dijkstra algorithm is $O(n \log n)$ in average

- Plan for 2023
  - Linear search of a maximal element (according to some partial order) in input stream of m elements has in average $O(\log m)$ flips of the leader. (Thanks to Eduard Lerner for recalling this statement!)
  - “Input stream” of all possible couplings consists of $n!$ individual couplings, while according to Stirling's approximation, $\ln n! = n \ln n - n + O(\ln n)$.
- Finally, what is the complexity of Dijkstra's assignment algorithm in the hardest case?