Expressive power of the partial fixed point operator for finite and infinite structures

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Relational database

- Codd E. F. A relational model for large shared data banks. // Communications of the ACM — 1970 — Vol. 13 — P. 377–387.
- Codd E. F. Relational completeness of data base sublanguages. // Database Systems (ed. Rustin R.) — Prentice-Hall — 1972 — P. 33–64.

First order logic

- Aho A., Ullman J. The universality of data retrieval languages. // In Proc. ACM Symp. on Principles of Programming Languages. — 1979. ACM Press — P. 110–120.

Extensions

- second order logic
- transitive closure operator
- fixed point operators

Introduction

Fixed point operators

- least fixed point (LFP),
- inflationary fixed point (IFP),
- partial fixed point (PFP).

Extension

Aho A., Ullman J. The universality of data retrieval languages. // In Proc. ACM Symp. on Principles of Programming Languages. — 1979. ACM Press — P. 110–120.

First order PFP-logic formula

A partial fixed point logic formula is defined like a first order logic formula and with the partial fixed point operator PFP. Let $\varphi(\bar{x}, \bar{y})$ be a formula that contains a non-language predicate symbol Q. Here, the length of $\bar{y}$ must be equal to the arity of Q. Then, $\mathrm{PFP}_{Q(\bar{y})}(\varphi)$ is a formula of an original language, this formula also contains two tuples of free variables $\bar{x}$ and $\bar{y}$.

Value of PFP-operator for finite structures

Let $\mathfrak{A}$ be a structure, and $\varphi(\bar{x}, \bar{y})$ be a formula that contains a new predicate symbol Q, where $\bar{x}$ and $\bar{y}$ are tuples of variables. Let us fix values of the variables $\bar{x}$ as $\bar{d} \in |\mathfrak{A}|$. The value of the formula $\mathrm{PFP}_{Q(\bar{y})}(\varphi)$ is defined as it is described in the following. Let us construct the sequence of sets

$$Q_0^{\bar{d}} = \varnothing \ \text{ and } \ Q_{i+1}^{\bar{d}} = \{\bar{y} \in |\mathfrak{A}| \mid (\mathfrak{A}, Q_i^{\bar{d}}) \models \varphi(\bar{d}, \bar{y})\}$$

for $i \in \omega$.

Definitions (finite structures)

Value of PFP-operator for finite structures

The value of partial fixed point is

$$\mathrm{PFP}_{Q(\bar{y})}(\varphi) = \begin{cases} Q_n^{\bar{d}} & \text{if } \; Q_n^{\bar{d}} = Q_{n+1}^{\bar{d}} \text{ for some } n, \\ \varnothing & \text{if } \; Q_n^{\bar{d}} \neq Q_{n+1}^{\bar{d}} \text{ for all } n. \end{cases}$$

In first case we say that operator $\mathrm{PFP}_{Q(\bar{y})}$ stabilizes in n steps. The value $Q_*^{\bar{d}}$ is $Q_n^{\bar{d}}$. In this case $\mathrm{PFP}_{Q(\bar{y})}(\varphi)(\bar{d}, \bar{y})$ is true for all $\bar{y} \in Q_*^{\bar{d}}$ and false for all $\bar{y} \notin Q_*^{\bar{d}}$.

In second case we say that operator $\mathrm{PFP}_{Q(\bar{y})}$ does not stabilize or fail in loop. In this case the value of $Q_*^{\bar{d}}$ is $\varnothing$ and formula is false for all $\bar{y}$.

Value of IFP-operator

Let $\mathfrak{A}$ be a structure, and $\varphi(\bar{x}, \bar{y})$ be a formula with a new predicate symbol Q, where $\bar{y}$ is a tuple of m elements. Let us fix the values of the variables $\bar{x}$ as $\bar{d} \in |\mathfrak{A}|$. A partial fixed point operator $\mathrm{IFP}_{Q(\bar{y})}(\varphi)(\bar{d})$ is the set $Q_*^{\bar{d}}$ constructed as follows. Let

$$Q_0^{\bar{d}} = \varnothing; \quad Q_{i+1}^{\bar{d}} = Q_i^{\bar{d}} \cup \{\bar{y} \in |\mathfrak{A}| \mid (\mathfrak{A}, Q_i^{\bar{d}}) \models \varphi(\bar{d}, \bar{y})\},$$

for all $i \in \omega$.

$$Q_*^{\bar{d}} = \bigcup_{i \in \omega} Q_i^{\bar{d}}$$

Introduction

PostgreSQL

```
WITH RECURSIVE cte_name (column1, column2, ...) AS(
   SELECT select_list FROM table1 WHERE condition

   UNION [ALL]

   SELECT select_list FROM cte_name WHERE rec_condition
)
SELECT * FROM cte_name;
```

Partial fixed point

Gurevich Y., Shelah S. Fixed-point extensions of first-order logic. // Annals of Pure and Applied Logic — 1986 — P. 265–280.

Finite structures

Libkin L. Elements of Finite Model Theory. — Berlin: Springer, 2004. — 314 p.

Infinite structures

Kreutzer S. Partial Fixed-Point Logic on Infinite Structure // Computer Science Logic. — 2002. — P. 337–351.

Definitions

PFP-operator semantic for infinite structures

- $\mathrm{PFP}^{\forall}$-operator
- $\mathrm{PFP}^{\exists}$-operator
- PFP^{Q}-quantifier

Definitions

$\mathrm{PFP}^{\forall}$-operator

The value of the partial fixed point $\mathrm{PFP}^{\forall}$ is the following set $Q_{\forall}^{\bar{d}}$. A tuple $\bar{y}$ belongs to the set $Q_{\forall}^{\bar{d}}$ if and only if the formula $Q_j(\bar{y})$ is true for almost every j. In the other words, there is some natural number i such that the formula $Q_j(\bar{y})$ is true for all natural numbers $j > i$. Therefore, for these $\bar{y}$ the formula $\mathrm{PFP}^{\forall}_{Q(\bar{y})}(\varphi)(\bar{d}, \bar{y})$ is true.

$\mathrm{PFP}^{\exists}$-operator

The value of the partial fixed point $\mathrm{PFP}^{\exists}$ is the following set $Q_{\exists}^{\bar{d}}$. A tuple $\bar{y}$ belongs to the set $Q_{\exists}^{\bar{d}}$ if and only if the tuple $\bar{y}$ belongs to sets $Q_i^{\bar{d}}$ infinitely often. That is, there are infinitely many i such that the formula $Q_i^{\bar{d}}(\bar{y})$ is true. Therefore, for these $\bar{y}$ the formula $\mathrm{PFP}^{\exists}_{Q(\bar{y})}(\varphi)(\bar{d}, \bar{y})$ is true.

$\mathrm{PFP}^{\forall}$-operator

Let us consider the finite graph $G = (V, E)$, where V is a set of vertices of the graph G and E is a set of edges. We consider this graph G as a structure, where V is the domain and $E^{(2)}$ is the unique binary predicate symbol. The formula $E(x, y)$ means that there is an edge from the vertex x to the vertex y. Then, the formula $\mathrm{PFP}^{\forall}_{Q(x)}(\theta)(v, w)$ is true if and only if the vertex w is reachable from the vertex v, where

$$\theta(v, x) \equiv x = v \vee Q(x) \vee (\exists y)\big(Q(y) \wedge E(y, x)\big).$$

Examples

$\mathrm{PFP}^{\exists}$-operator

The formula $(\exists a)(\exists b)\,\mathrm{PFP}^{\exists}_{Q(u,x)}(\theta)(a,v,w)$ is true if and only if there are paths of unbounded lengths from the vertex v to the vertex w, where

$$
\begin{aligned}
\theta(u,a,b,v,x) \equiv a \neq b \wedge \\
&\big(\neg(\exists y)Q(a,y) \wedge \neg(\exists y)Q(b,y) \rightarrow u = a \wedge x = v\big) \wedge \\
&\big((\exists y)Q(a,y) \wedge (\exists z)(\exists y)\big(Q(a,y) \wedge E(y,z)\big) \rightarrow \\
&\qquad u = a \wedge (\exists y)\big(Q(a,y) \wedge E(y,x)\big)\big) \wedge \\
&\big((\exists y)Q(a,y) \wedge \neg(\exists z)(\exists y)\big(Q(a,y) \wedge E(y,z)\big) \rightarrow u = b\big) \wedge \\
&\big(\neg(\exists y)Q(a,y) \wedge (\exists y)Q(b,y) \rightarrow u = b\big).
\end{aligned}
$$

Expressive power of $\mathrm{PFP}^{\forall}$- and $\mathrm{PFP}^{\exists}$-operators

Theorem

Let φ be an arbitrary formula. Then, the formula $\mathrm{PFP}^{\forall}_{Q(\bar{y})}(\varphi)(\bar{y})$ is equivalent to the formula

$$\mathrm{PFP}^{\exists}_{Q(\bar{y})}(\varphi)(\bar{y}) \wedge \neg(\exists a)(\exists b)\,\mathrm{PFP}^{\exists}_{P(u,\bar{y})}(\theta)(b,a,b,\bar{y})$$

for all tuples $\bar{y}$.

Formula $\theta(u,a,b,\bar{y})$ does not contain new $\mathrm{PFP}^{\forall}$-operators.

Formula θ

$$\theta(u,a,b,\bar{y}) \equiv a \neq b \wedge \\ [u = a \wedge \varphi'(a,\bar{y}) \vee u = b \wedge \neg\varphi'(a,\bar{y}) \wedge P(a,\bar{y})].$$

Expressive power of $\mathrm{PFP}^{\forall}$- and $\mathrm{PFP}^{\exists}$-operators

Formula φ'

We introduce a new predicate symbol P. The arity of the predicate P is $w+1$, where w is the arity of the predicate Q.

$$\varphi'(a,\bar{y}) \equiv (\varphi)^{Q(\bar{t})}_{P(a,\bar{t})}(\bar{y}).$$

Here in the formula φ we replace each occurrence $Q(\bar{t})$ with $P(a,\bar{t})$ for all tuples $\bar{t}$.

The first argument of the predicate P will be used as follows: $P_i(a,\bar{y})$ is equivalent to $Q_i(\bar{y})$, and $P_i(b,\bar{y})$ means that the tuple $\bar{y}$ is added in the previous step, but is missing in the current step.

Theorem proof. $\mathrm{PFP}^{\forall}_{Q(\bar{y})}(\varphi)(\bar{c}) \Leftrightarrow$

$$\mathrm{PFP}^{\exists}_{Q(\bar{y})}(\varphi)(\bar{y}) \land \neg(\exists a)(\exists b)\,\mathrm{PFP}^{\exists}_{P(u,\bar{y})}(\theta)(b,a,b,\bar{c}).$$

Expressive power of $\mathrm{PFP}^{\forall}$- and $\mathrm{PFP}^{\exists}$-operators

Theorem

Let φ be an arbitrary formula. Then, the formula $\mathrm{PFP}^{\exists}_{Q(\bar{y})}(\varphi)(\bar{y})$ is equivalent to the formula $\neg(\exists a)(\exists b)\,\mathrm{PFP}^{\forall}_{P(u,\bar{y})}(\theta)(b, a, b, \bar{y})$.

Formula $\theta(u, a, b, \bar{y})$ does not contain new $\mathrm{PFP}^{\exists}$-operators.

Formula θ

$$\theta(u, a, b, \bar{y}) \equiv a \neq b \wedge [u = a \wedge \varphi'(a, \bar{y}) \vee u = b \wedge \neg\varphi'(a, \bar{y})].$$

Expressive power of $\mathrm{PFP}^{\forall}$- and $\mathrm{PFP}^{\exists}$-operators

Formula φ'

$$\varphi'(a, \bar{y}) \equiv (\varphi)^{Q(\bar{t})}_{P(a,\bar{t})}(\bar{y})$$

The first argument of the predicate P is used as follows: $P_i(a, \bar{y})$ is equivalent to $Q_i(\bar{y})$, and $P_i(b, \bar{y})$ is constructed as a complement of the predicate Q_i at any step.

Theorem proof.

$$\mathrm{PFP}^{\exists}_{Q(\bar{y})}(\varphi)(\bar{c}) \Leftrightarrow \neg(\exists a)(\exists b)\,\mathrm{PFP}^{\forall}_{P(u,\bar{y})}(\theta)(b, a, b, \bar{c}).$$

Expressive power of $\mathrm{PFP}^{\forall}$- and $\mathrm{PFP}^{\exists}$-operators

Corollary

$\mathrm{PFP}^{\forall}$-logic and $\mathrm{PFP}^{\exists}$-logic have the same expressive power.

Definitions

PFP^Q-quantifier

The formula $\left([\mathrm{PFP}^Q_{Q(\bar{y})}(\psi)]\varphi\right)(\bar{d},\bar{y})$ is satisfied if and only if there is a natural number i such that

$$Q_i^{\bar{d}} = \{\bar{y} \in |\mathfrak{A}| \mid \mathfrak{A} \models \varphi(\bar{d},\bar{y})\}.$$

Examples

PFP^{Q}-quantifier

The formula $\left[\mathrm{PFP}^{Q}_{Q(x)}(\theta)\right](x = w)$ is true if and only if the vertex w is the unique vertex located at some distance from the vertex v, where

$$\theta(v, x) \equiv \left(\neg(\exists y)Q(y) \rightarrow x = v\right) \wedge \left((\exists y)Q(y) \rightarrow (\exists y)(Q(y) \wedge E(y, x))\right).$$

Expressive power of $\mathrm{PFP}^{\forall}$-operator and PFP^{Q}-quantifier

Theorem

Let $\varphi(\bar{x},\bar{y})$ and $\psi(\bar{x},\bar{y})$ be arbitrary formulas, where the formula ψ contains the predicate symbol Q, and a length of the tuple $\bar{y}$ is the arity of Q. Then, the formula

$$\left[\mathrm{PFP}^{Q}_{Q(\bar{y})}(\psi)\right]\varphi$$

is equivalent to the formula

$$(\forall\bar{y})\big((\exists a,b)\,\mathrm{PFP}^{\forall}_{P(u,\bar{y})}(\theta)(a,a,b,\bar{y})\leftrightarrow\varphi(\bar{y})\big).$$

Expressive power of $\mathrm{PFP}^{\forall}$-operator and PFP^{Q}-quantifier

Formula θ

The formula θ defines the predicate P value as follows. Alternating the first element a or b, tuples corresponding to the set Q are added until we obtain the set given by the formula φ. After such a set has been obtained, at each step we save its tuples with the first element a.

Theorem proof.

$$\left(\mathrm{PFP}^{Q}_{Q(\bar{y})}(\psi)\right)\varphi \Leftrightarrow (\forall \bar{y})\left((\exists a, b)\,\mathrm{PFP}^{\forall}_{P(u,\bar{y})}(\theta)(a, a, b, \bar{y}) \leftrightarrow \varphi(\bar{y})\right).$$

Expressive power of $\mathrm{PFP}^{\forall}$-operator and PFP^{Q}-quantifier

Theorem

Let $\varphi(\bar{x}, \bar{y})$ be an arbitrary formula, where the formula φ contains the predicate symbol Q, and a length of the tuple $\bar{y}$ is the arity of Q. Then, the formula

$$\mathrm{PFP}^{\forall}_{Q(\bar{y})}(\varphi)(\bar{y})$$

is equivalent to the formula

$$(\exists a, b, c)\left[\mathrm{PFP}^{Q}_{P(u,\bar{z})}(\theta_1)\right](u = c).$$

The formula $\theta_1(u, a, b, c, \bar{y})$ does not contains new $\mathrm{PFP}^{\forall}$-operators.

Expressive power of $\mathrm{PFP}^{\forall}$-operator and PFP^{Q}-quantifier

Formula θ_1

We add to the set P tuples of the form $(a, \bar{y})$ where $\bar{y} \in Q_i$. First we reach the step where the tuple $\bar{y}$ is added along with a, we mark this by adding all possible tuples of the form $(b, \bar{y})$. If at least in one of the next steps the tuple $\bar{y}$ disappears, which is checked by the internal PFP-operator, then we continue further construction. Otherwise, we add all possible tuples of the form $(c, \bar{y})$ to mark finding a step, starting from which $\bar{y}$ will belong to all sets Q.

Theorem proof.

$$\mathrm{PFP}^{\forall}_{Q(\bar{y})}(\varphi)(\bar{y}) \Leftrightarrow (\exists a, b, c)(\mathrm{PFP}^{Q}_{P(u,\bar{z})}(\theta_1))(u = c).$$

Expressive power of $\mathrm{PFP}^{\forall}$-operator and PFP^{Q}-quantifier

Corollary

$\mathrm{PFP}^{\forall}$-logic and PFP^{Q}-quantifier logic have the same expressive power.

Theorem

If theory T is ω-categorical then $\mathrm{PFP}^{\forall}$-operator can be eliminated in the theory T.

Proof. Let us denote all nonequivalent formulas by ψ_i, and the number of such formulas by m. We can eliminate the $\mathrm{PFP}^{\forall}$-operator as follows:

$$\bigvee_{j_0=1}^{m} \dots \bigvee_{j_m=1}^{m} \Big[(\forall \bar{u})\big(\psi_{j_0}(\bar{u}) \leftrightarrow (\varphi)^{Q}_{\neg\top}(\bar{u})\big) \wedge \bigwedge_{t}^{m-1} (\forall \bar{u})\big(\psi_{j_{t+1}}(\bar{u}) \leftrightarrow (\varphi)^{Q}_{\psi_{j_t}}(\bar{u})\big) \wedge$$
$$\wedge \Big(\bigvee_{l=0}^{m} \bigvee_{k>l}^{m} \big[(\forall \bar{u})\big(\psi_{j_l}(\bar{u}) \leftrightarrow \psi_{j_k}(\bar{u})\big) \wedge [\bigwedge_{p=l}^{k} \psi_{j_p}(x_1, \dots, x_s, y_1, \dots, y_n)]\big]\Big)\Big].$$

PFP-operator for finite structures

Theorem for first order

If the structure is finite and has a linear order relation, then for any PFP-logic formula φ with first-order quantifiers we can construct an equivalent formula of the form

$$(M\bar{y}_1)\,\mathrm{PFP}_{Q(\bar{y}_2)}(\psi),$$

where M are first-order quantifiers, and ψ is a first-order logic formula.

Proof

Theorem proof. If the formula φ does not contain any partial fixed point operator, then a dummy PFP-operator can be added to it.

To prove this, we use induction on the construction of the formula φ.

PFP-operator for finite structures

Theorem for second order

If the structure is finite and has a relation of linear order, then for any formula of PFP-logic with first- and second-order quantifiers one can construct an equivalent formula of the form

$$(M\bar{y}_1)\,\mathrm{PFP}_{Q(\bar{y}_2)}(\psi),$$

where M is a first-order quantifier prefix, and ψ is a first-order logic formula.

Proof

Theorem proof. We replace each subformula of the form $(\exists Q)\psi$ with $(\psi)^{Q}_{\mathrm{PFP}_{Q(\bar{x})}(\chi_\exists)}$, and $(\forall Q)\psi$ with $(\psi)^{Q}_{\mathrm{PFP}_{Q(\bar{x})}(\chi_\forall)}$, where $(\exists Q)$ and $(\forall Q)$ — second order quantifiers. Let us define the formulas $\chi_\exists$ and $\chi_\forall$:

$$\begin{aligned}\chi_\exists \equiv\ & \psi \wedge Q(\bar{x}) \vee \\ & \neg\psi \wedge (\exists \bar{x})\,\mathrm{follow}^{Q}(\bar{x}) \wedge \mathrm{follow}^{Q}(\bar{x}) \vee \\ & \neg\psi \wedge \neg(\exists \bar{y})\neg Q(\bar{y}) \wedge Q(\bar{x}),\end{aligned}$$

$$\begin{aligned}\chi_\forall \equiv\ & \neg\psi \wedge Q(\bar{x}) \vee \\ & \psi \wedge (\exists \bar{x})\,\mathrm{follow}^{Q}(\bar{x}) \wedge \mathrm{follow}^{Q}(\bar{x}) \vee \\ & \psi \wedge \neg(\exists \bar{y})\neg Q(\bar{y}) \wedge Q(\bar{x}).\end{aligned}$$

Expressive power of PFP- and IFP-operators

Preorder

A non-strict preorder is a binary relation on a set that is reflexive and transitive.

Preorder leq

Let a structure have a strict partial order relation $<$, in which there are arbitrarily long discrete chains. Then the preorder leq is discrete and it contains an infinite discrete chain of consecutive elements.

$$\mathrm{leq}(\bar{n}_1, a, b, \bar{n}_2, c, d) = \mathrm{IFP}_{L(\bar{m},u,v)}(\theta_L)(\bar{n}_1, a, b, \bar{n}_2, c, d)$$

Expressive power of PFP- and IFP-operators

Formula Ψ

For an arbitrary operator $\mathrm{PFP}_{Q(\bar{y})}(\varphi)$, we can construct the formula Ψ, which is an IFP-operator:

$$\Psi \equiv \mathrm{IFP}_{P(\bar{n},u,v,\bar{y})}(\theta).$$

Expressive power of PFP- and IFP-operators

Theorem

The formula $\mathrm{PFP}_{Q(\bar{x})}(\varphi)(\bar{z})$ is satisfied if and only if the formula $F(\bar{z})$ is satisfied, given as follows:

$$
\begin{aligned}
F(\bar{z}) \equiv\ & (\exists \bar{n}_1)(\exists a)(\exists b)(\exists \bar{n}_2)(\exists c)(\exists d)\big(\\
& \quad \mathrm{leq}(\bar{n}_1, a, b, \bar{n}_2, c, d) \wedge \neg\, \mathrm{leq}(\bar{n}_2, c, d, \bar{n}_1, a, b) \wedge \\
& \wedge (\forall \bar{m})(\forall s)(\forall t)(\mathrm{leq}(\bar{n}_1, a, b, \bar{m}, s, t) \wedge \neg\, \mathrm{leq}(\bar{m}, s, t, \bar{n}_1, a, b) \rightarrow \\
& \qquad \rightarrow \mathrm{leq}(\bar{n}_2, c, d, \bar{m}, s, t)) \wedge \\
& \wedge (\forall \bar{y})(\Psi(\bar{n}_1, a, b, \bar{y}) \leftrightarrow \Psi(\bar{n}_2, c, d, \bar{y})) \wedge \Psi(\bar{n}_1, a, b, \bar{z})\big).
\end{aligned}
$$

Expressive power of PFP- and IFP-operators

Theorem

The formula $\mathrm{PFP}^{\forall}_{Q(\bar{x})}(\varphi)(\bar{z})$ is satisfied if and only if the formula $F^{\forall}(\bar{z})$ is satisfied, given as follows:

$$F^{\forall}(\bar{z}) \equiv (\exists \bar{n}_1)(\exists a)(\exists b)\big(\eta(\bar{n}_1, a, b) \wedge$$
$$(\forall \bar{n}_2)(\forall c)(\forall d)(\mathrm{leq}(\bar{n}_1, a, b, \bar{n}_2, c, d) \to \Psi(\bar{n}_2, c, d, \bar{z}))\big).$$

Auxiliary formula

The pair (a, b) belongs to some infinite discrete chain:

$$\eta(\bar{n}_1, a, b) \equiv (\forall \bar{m})(\forall e)(\forall f)(\mathrm{leq}(\bar{n}_1, a, b, \bar{m}, e, f) \to$$
$$\to (\forall \bar{m}')(\exists e')(\exists f')(\mathrm{leq}(\bar{m}, e, f, \bar{m}', e', f') \wedge \neg\, \mathrm{leq}(\bar{m}', e', f', \bar{m}, e, f))).$$

Expressive power of PFP- and IFP-operators

Theorem

The formula $\mathrm{PFP}^{\exists}_{Q(\bar{x})}(\varphi)(\bar{z})$ is satisfied if and only if the formula $F^{\exists}(\bar{z})$ is satisfied, given as follows:

$$F^{\exists}(\bar{z}) \equiv F^{\exists}_1(\bar{z}) \vee F^{\exists}_2(\bar{z}),$$

where

$$\begin{aligned}
F^{\exists}_1(\bar{z}) \equiv\ & (\exists \bar{n}_1)(\exists a)(\exists b)\big(\neg a = b \wedge \mathrm{leq}(\bar{n}_1, a, a, \bar{n}_1, a, b) \wedge \\
& \wedge \neg(\exists \bar{x})\Psi(\bar{n}_1, a, b, \bar{x})\big) \wedge (\exists \bar{n}_1)(\exists a)(\exists b)\Psi(\bar{n}_1, a, b, \bar{z}), \\
F^{\exists}_2(\bar{z}) \equiv\ & (\forall \bar{n}_1)(\forall a)(\forall b)\big(\eta(\bar{n}_1, a, b) \rightarrow \\
& \rightarrow (\exists \bar{n}_2)(\exists c)(\exists d)(\mathrm{leq}(\bar{n}_1, a, b, \bar{n}_2, c, d) \wedge \Psi(\bar{n}_2, c, d, \bar{z}))\big).
\end{aligned}$$

Expressive power of PFP- and IFP-operators

Consequence

The partial fixed point operators of the semantics PFP, $\mathrm{PFP}^{\forall}$ and $\mathrm{PFP}^{\exists}$ can be modelled using the inflationary fixed point operator.

Decidability of $\mathrm{PFP}^{\forall}$-operator

Theorem

We consider the structure $(\mathbb{Z}, s^{(1)})$. Here $\mathbb{Z}$ is the set of integers, and $s^{(1)}$ is the successor function. The halting problem for Minsky machines with two counters is reducible to the truth problem in the structure $(\mathbb{Z}, s^{(1)})$ for a partial fixed point logic formulas. Moreover, formulas contain a unique unary PFP-operator.

Proof

It is well known that we can assume the following.

1. the states are numbered from 0 to n, and the commands are numbered from 0 to $m - 1$,
2. among the states there is only one final, and it has the number n,
3. the values of both counters are 0 when the Minsky machine $\mathfrak{M}$ stops.

Proof

The configuration of the Minsky machine with two counters has the form (q_i, g, h), where $q_i \in S$ is the state, g is the value of the first counter, and h is the value of the second counter.

A unary relation E on the set of integers can be represented as a sequence of 0 and 1. In this representation, 1 denotes that the corresponding number belongs to the set E, and 0 denotes the opposite. We encode the configuration (q_i, g, h) of the Minsky machine $\mathfrak{N}$ by the unary relation as follows:

$$\ldots \overset{a}{0}\, 1 \underbrace{0 \ldots \overset{b}{0}}_{g+1} \overset{z}{1}\, 1 \underbrace{\underbrace{0 \ldots \overset{c}{0}}_{i+1} 111 \underbrace{0 \ldots \overset{d}{0}}_{n-i+1}}_{n+5} 1111 \underbrace{0 \ldots \overset{e}{0}}_{h+1} 111110 \ldots$$

Proof

For each of the four types of Minsky machine commands, we construct the corresponding formula.

For example, for the command $q_i \to inc_1, q_j$ the formula construct as follows:

$$\begin{aligned}(\forall a, b, c, d, e)\big((\exists u)Q(u) \wedge \varphi_i(a, b, c, d, e) \to \psi(x, b, d) \vee \\ x = a \vee \\ x = s^{3+j+1}b \vee x = s^{3+j+2}b \vee x = s^{3+j+3}b \vee \\ x = se \vee x = s^2 e \vee x = s^3 e \vee x = s^4 e \vee x = s^5 e\big).\end{aligned}$$

Proof

Let us construct the following formula for the logic of a partial fixed point:

$$(\exists z)(\forall u)\big[\,\mathrm{PFP}^{\forall}_{Q(x)}\,\big(\theta(z,x)\big)(u) \leftrightarrow \tau(z,u)\big],$$

where

$$\begin{aligned}\theta(z,x) \equiv & [\neg Q(z) \rightarrow \alpha(z,x)] \wedge \\ & [\bigwedge_{l=0}^{m-1} p_l] \wedge \\ & [\tau^{Q}(z) \rightarrow Q(x)].\end{aligned}$$

Undecidability of the $\mathrm{PFP}^{\forall}$-operator

Comment

It is easy to see that the formula θ under the partial fixed point operator is equivalent to some universal formula since all existence quantifiers are in premises of implications.

Corollary

The halting problem for a Minsky machine with two counters is reducible to the truth problem for a PFP-logic formula in the algebraic structure $(\mathbb{Z}, s^{(1)})$. Moreover, all PFP-operators are non-nested, unary, and applied only to universal formulas.

Undecidability of the $\mathrm{PFP}^{\forall}$-operator

Theorem

We consider the structure $(\mathbb{Z}, s^{(1)})$. Here $\mathbb{Z}$ is the set of integers, and $s^{(1)}$ is the successor function that has the first element. The halting problem for Minsky machines with two counters is reducible to the truth problem in the structure $(\mathbb{Z}, s^{(1)})$ for a partial fixed point logic formulas. Moreover, formulas contain a unique unary PFP-operator.

Let us show how to construct the corresponding partial fixed point formula:

$$(\exists o)\Big[\neg(\exists p)(sp = o) \wedge (\exists z)(\forall u)\big[\,\mathrm{PFP}^{\forall}_{Q(x)}\,\big(\theta(o,z,x)\big)(u) \leftrightarrow \tau(z,u)\big]\Big],$$

where

$$\begin{aligned}\theta(o,z,x) \equiv & [\neg Q(z) \rightarrow \alpha(z,x)] \wedge \\ & [\bigwedge_{l=0}^{m-1} p_l] \wedge \\ & [\tau^{Q}(z) \rightarrow Q(x)] \wedge \\ & [Q(o) \rightarrow sx = x].\end{aligned}$$

Possible questions

1. In some theorems, the constructed formulas use nested fixed point operators. Is it possible to limit ourselves to a single non-nested operator for infinite structures?
2. In many theorems, the constructed formulas use partial fixed point operators whose arity is more than the original ones. It is known that unary and binary inflationary fixed point operators have different properties. Therefore, the question arises about the possibility of constructing PFP-operators without increasing the arity.
3. Determine which second order logic formulas can be converted to partial fixed point formulas for infinite structures.

Thanks for your attention