Mezhirov's game for intuitionistic logic and its variations

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Basic definitions (propositional case)

Let Ω be the propositional intuitionistic language with $\perp$ and the set of logical connectives $\{\to, \wedge, \vee\}$, where $\neg A$ will be considered as $A \to \perp$.

Definition

For the set of formulas Γ let $\mathcal{F}(\Gamma) \leftrightharpoons \{\psi | \psi$ is a subformula of some formula from $\Gamma\}$.

Definition

A position in the game $\mathcal{C} \leftrightharpoons \langle \mathcal{O}, \mathcal{P} \rangle$, where $\Gamma \subseteq \mathcal{O} \cup \mathcal{P} \subseteq \mathcal{F}(\Gamma)$ (in all our games will also be satisfied a condition $|\Gamma \cap \mathcal{P}| = 1$) .
A starting position is a $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\} \rangle$, where $\Gamma = \mathcal{O}_0 \cup \{\varphi\}$.

Truth relation (propositional case)

Definition

The truth relation $\Vdash$ for the position $\mathcal{C} = \langle \mathcal{O}, \mathcal{P} \rangle$ and formulas from $\mathcal{F}(\Gamma)$ is defined recursively by the following rules:

$$\mathcal{C} \nVdash \bot$$
$$\mathcal{C} \Vdash \mathsf{p} \rightleftharpoons \mathsf{p} \in \mathcal{O} \text{ for } \mathsf{p} \in \mathsf{Prop}$$
$$\mathcal{C} \Vdash \varphi \star \psi \rightleftharpoons \varphi \star \psi \in \mathcal{O} \cup \mathcal{P} \text{ and } (\mathcal{C} \Vdash \varphi) \star (\mathcal{C} \Vdash \psi) \text{ for } \star \in \{\to, \wedge, \vee\}$$

Order of turns

Let us call a formula from $\mathcal{P}$ Proponent's mistake if it is false in the current position (the same for $\mathcal{O}$ and Opponent). If Opponent has no mistakes but Proponent has, then Proponent moves. Otherwise, Opponent must move.

Player's turn (propositional case)

During the turn

Proponent in his turn in the position $\mathcal{C} = \langle \mathcal{O}, \mathcal{P} \rangle$ can expand $\mathcal{P}$ by adding one formula from $\mathcal{F}(\Gamma)$.
Opponent in his turn in the position $\mathcal{C} = \langle \mathcal{O}, \mathcal{P} \rangle$ can expand $\mathcal{O}$ by adding one formula from $\mathcal{F}(\Gamma)$.

Conditions for winning

A player loses if he cannot make a move.

Example 1 (propositional case)

$\Phi = (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$
ant $= \neg q \rightarrow \neg p$
con $= p \rightarrow q$

	Position 2			Position 3			Position 4			Position 5			Position 6		
	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$
Φ		×	0		×	1		×	0		×	1		×	1
ant	×		1	×		1	×		1	×		0	×		1
con			0		×	1		×	0		×	0		×	1
$\neg q$			0			0			0		×	1		×	0
$\neg p$			0			0			0			0			0
q			0			0			0			0	×		1
p			0			0	×		1	×		1	×		1

Example 2 (propositional case)

$\Phi = \neg\neg\neg p \rightarrow \neg p$

	Position 1			Position 2			Position 3			Position 4		
	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$	$\mathcal{O}$	$\mathcal{P}$	$\Vdash$
Φ		$\times$	1		$\times$	0		$\times$	1		$\times$	1
$\neg\neg\neg p$			0	$\times$		1	$\times$		1	$\times$		0
$\neg\neg p$			0			0			0			1
$\neg p$			0			0		$\times$	1		$\times$	0
p			0			0			0	$\times$		1

Main result (propositional case)

Theorem

Proponent has a winning strategy in position $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\} \rangle$ (with only finite $\mathcal{O}_0$) iff $\mathcal{O}_0 \vDash \varphi$, where $\vDash$ is the entailment in propositional inuitionistic logic.

Basic definitions

Let $\Sigma = \langle Pred, Const, arity \rangle$ be our signature (without function symbols). Then let Ω be the elementary intuitionistic language of this signature; language will contain $\bot$, and the set of logical connectives will be $\{\rightarrow, \wedge, \vee\}$, where $\neg A$ will be considered as $A \rightarrow \bot$.

Definition

For the set of formulas Γ and set of objects (constants) Δ let $\mathcal{F}(\Gamma, \Delta) \leftrightharpoons \{P[c_1, \ldots, c_n] | P[x_1, \ldots, x_n]$ is a subformula of some formula from Γ and free variables of it are only $x_1, \ldots, x_n; c_i \in \Delta\}$.

Definition

A position in the game $\mathcal{C} \leftrightharpoons \langle \mathcal{O}, \mathcal{P}, \Delta, \Gamma \rangle$, where $\Gamma \subseteq \mathcal{O} \cup \mathcal{P} \subseteq \mathcal{F}(\Gamma, \Delta)$ (in all our games will also be satisfied a condition $|\Gamma \cap \mathcal{P}| = 1$) .
A starting position is a $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\}, \Delta_0, \Gamma \rangle$, where $\Gamma = \mathcal{O}_0 \cup \{\varphi\}$ and Δ_0 is an exact set of all constants contained in formulas from the set Γ.

Truth relation

Definition

The truth relation $\Vdash$ for the position $\mathcal{C} = \langle \mathcal{O}, \mathcal{P}, \Delta, \Gamma \rangle$ and formulas from $\mathcal{F}(\Gamma, \Delta)$ is defined recursively by the following rules:

$$\mathcal{C} \nVdash \bot$$
$$\mathcal{C} \Vdash A[c_1, \dots, c_n] \rightleftharpoons A[c_1, \dots, c_n] \in \mathcal{O} \text{ for } A \in \text{Pred, arity}(A) = n,\ c_i \in \Delta$$
$$\mathcal{C} \Vdash \varphi \star \psi \rightleftharpoons \varphi \star \psi \in \mathcal{O} \cup \mathcal{P} \text{ and } (\mathcal{C} \Vdash \varphi) \star (\mathcal{C} \Vdash \psi) \text{ for } \star \in \{\rightarrow, \wedge, \vee\}$$
$$\mathcal{C} \Vdash qxP[x] \rightleftharpoons qxP[x] \in \mathcal{O} \cup \mathcal{P} \text{ and } q\alpha \in \Delta(\mathcal{C} \Vdash P[\alpha]) \text{ for } q \in \{\exists, \forall\}$$

Order of turns

Let us call a formula from $\mathcal{P}$ Proponent's mistake if it is false in the current position (the same for $\mathcal{O}$ and Opponent). If Opponent has no mistakes but Proponent has, then Proponent moves. Otherwise, Opponent must move.

Player's turn

During the turn

Proponent in his turn in the position $\mathcal{C} = \langle \mathcal{O}, \mathcal{P}, \Delta, \Gamma \rangle$ can expand $\mathcal{P}$ by adding formulas from $\mathcal{F}(\Gamma, \Delta)$.
Opponent in his turn in the position $\mathcal{C} = \langle \mathcal{O}, \mathcal{P}, \Delta, \Gamma \rangle$ can expand Δ by adding any new elements and also can expand $\mathcal{O}$ by adding formulas from $\mathcal{F}(\Gamma, \Delta)$.

Conditions for winning

A player loses if he cannot pass turn to another player (so if he make a move twice in a row, he loses).
In an infinite game Proponent wins.

Example 1

	$\mathcal{O}$	$\mathcal{P}$	Δ
0	$\varnothing$	$\forall y \exists x(P[x] \to P[y])$	$\varnothing$
1	$\varnothing$	$\forall y \exists x(P[x] \to P[y])$	$\{\alpha\}$
2	$\varnothing$	$\forall y \exists x(P[x] \to P[y]), \exists x(P[x] \to P[\alpha]), P[\alpha] \to P[\alpha]$	$\{\alpha\}$
3	$\varnothing$	$\forall y \exists x(P[x] \to P[y]), \exists x(P[x] \to P[\alpha]), P[\alpha] \to P[\alpha]$	$\{\alpha, \beta\}$

Example 2

$\varphi = \forall x[(P[x] \to \forall x P[x]) \to \forall x P[x]] \to \forall x P[x]$
$\psi[x] = [(P[x] \to \forall x P[x]) \to \forall x P[x]]; (\varphi = \forall x \psi[x] \to \forall x P[x])$

	$\mathcal{O}$	$\mathcal{P}$	Δ
0	$\varnothing$	φ	$\varnothing$
1	$\forall x \psi[x]$	φ	$\varnothing$
2	$\forall x \psi[x]$	$\varphi, \forall x P[x]$	$\varnothing$
3	$\forall x \psi[x], \psi[\alpha]$	$\varphi, \forall x P[x]$	$\{\alpha\}$
4	$\forall x \psi[x], \psi[\alpha]$	$\varphi, \forall x P[x], P[\alpha] \to \forall x P[x]$	$\{\alpha\}$
5	$\forall x \psi[x], \psi[\alpha], P[\alpha], \psi[\beta]$	$\varphi, \forall x P[x], P[\alpha] \to \forall x P[x]$	$\{\alpha, \beta\}$

Main results

Theorem

Proponent has a winning strategy in position $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\}, \Delta_0, \mathcal{O}_0 \cup \{\varphi\} \rangle$ (with possibly infinite $\mathcal{O}_0$) iff $\mathcal{O}_0 \vDash \varphi$, where $\vDash$ is the entailment in logic of all Noetherian Kripke frames.

Theorem

Proponent has a winning strategy in position $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\}, \Delta_0, \mathcal{O}_0 \cup \{\varphi\} \rangle$ (with only finite $\mathcal{O}_0$) iff $\mathcal{O}_0 \vDash \varphi$, where $\vDash$ is the entailment in logic of all Casari's Kripke frames (class of all Kripke frames in which Casari's formula $\varphi = \forall x[(P[x] \to \forall x P[x]) \to \forall x P[x]] \to \forall x P[x]$ is valid; Kripke frame is from Casari's class iff in every countable sequence of worlds ω_i their individual domains Δ_i remain finite and stabilize).

Theorem

In the finite variation, Proponent has a winning strategy in position $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\}, \Delta_0, \mathcal{O}_0 \cup \{\varphi\} \rangle$ (with possibly infinite $\mathcal{O}_0$, but with only finite Δ_0) iff $\mathcal{O}_0 \vDash \varphi$, where $\vDash$ is the entailment in logic of all Noetherian Kripke frames with only finite individual domains Δ in each world.

Theorem

In the finite variation, Proponent has a winning strategy in position $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\}, \Delta_0, \mathcal{O}_0 \cup \{\varphi\} \rangle$ (with only finite $\mathcal{O}_0$) iff $\mathcal{O}_0 \vDash \varphi$, where $\vDash$ is the entailment in logic of all Casari's Kripke frames with only finite individual domains Δ in each world.

Theorem

In the finite variation, Proponent has a winning strategy in position $\mathcal{C}_0 = \langle \mathcal{O}_0, \{\varphi\}, \Delta_0, \mathcal{O}_0 \cup \{\varphi\} \rangle$ (with only finite $\mathcal{O}_0$) iff $\mathcal{O}_0 \vDash \varphi$, where $\vDash$ is the entailment in logic of all finite Kripke frames with only finite individual domains Δ in each world.

Link to preprint

https://arxiv.org/abs/2310.16206

Thank you!