### Impressions and results of «Interactions between Proof Assistants and Mathematics» Sep 2023, Regensburg

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### **ITP-2023 school**

- Two weeks Sep 18–29, 2023, Regensburg
- About 50 participants, mostly MS and PhD students (mostly mathematicians) • Lectures + contributed talks (by participants)
- Topics:
  - Proof assistants (Coq, Lean, Rzk, a little bit of Agda, a little bit of LISA)
  - Mathematics (most with formalization exercises):
    - HoTT,  $\infty$ -categories, synthetic algebraic geometry, algebraic type theory, modal type theory, algebraic automata theory
  - History and development of theorem provers (HOL Light)

- Lecture by Mike Shulman
- Great exposition
- Good intuition
- Provoked discussions about modalities for  $\infty$ -categories and adding modalities in Rzk
- Later, I found an interesting paper that builds further on this: <u>arxiv/2301.13780</u>

#### High level programming



#### Find Shulman's slides and Agda code at <u>itp-school-2023.github.io/program</u>



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#### High level mathematics





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#### The need for discontinuity

In classical mathematics, we have the Intermediate Value Theorem:

Theorem (in classical mathematics)

For any continuous function  $f : [a, b] \rightarrow \mathbb{R}$  and point c with f(a) < c < f(b), there exists  $x \in [a, b]$  with f(x) = c.

In synthetic topology, where all functions are continuous, we expect to drop the adjective:

Theorem? (in synthetic topology)

For any function  $f : [a, b] \to \mathbb{R}$  and point c with f(a) < c < f(b), there exists  $x \in [a, b]$  with f(x) = c.



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#### Discontinuity

Thus, in synthetic topology we have primitive notions of both (continuous) function and also discontinuous function.

The former form the usual function-types  $A \rightarrow B$  and  $(x:A) \rightarrow B$ ; the latter form a new type  $(x:^{\flat} A) \rightarrow B$ .

Theorem (in (one version of) synthetic topology)  $(f :^{\flat} [a, b] \rightarrow \mathbb{R}) \rightarrow (c :^{\flat} \mathbb{R}) \rightarrow (f(a) < c < f(b))$ 

 $\rightarrow \exists (x \in [a, b]). f(x) = c.$ 

I'll sketch a proof of this, after introducing more structure.

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#### Modal operators

We can reify discontinuous functions in two ways:

- (1)  $(x : {}^{\flat} A) \to B$  is equivalent to  $(x : {}^{\flat} A) \to B$ .
  - $\flat A$  is A "retopologized discretely".
  - $\flat$  is a coreflection into the subcategory of discrete types.
- 2  $(x : A) \to B$  is also equivalent to  $(x : A) \to \#B$ .
  - #B is B "retopologized indiscretely".
  - # is a reflection into the subcategory of indiscrete types.
- **3** It follows that  $\flat \dashv \ddagger$ .

Such unary type operators are called modalities, after the classical  $\Box$  ("It is necessary that...") and  $\Diamond$  ("It is possible that...") from modal logic.

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### ITP-2023 highlight #2: Lean

- Tutorials by Jannis Limperg
- Good examples
- Interactive sessions
- I was particularly interested in syntactic choices, such as calc mode

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alc Mode -/

{fg:  $\alpha \rightarrow \beta \rightarrow \gamma$ } (fg: f = g) (ab : a = b) (bc : b = c) : x = y) : f a x = g c y := byx = f b x := by rw [ab]= f c x := by rw [bc]= f c y := by rw [xy]= g c y := by rw [fg]

Find Limperg's material on GitHub at <u>JLimperg/regensburg-itp-school-2023</u>



### ITP-2023 highlight #3: Rzk and sHoTT (before)

- Rzk version 0.5.4
- 3 users (Emily, Jonathan, and Fredrik Bakke)
- Formalized the  $\infty$ -categorical Yoneda lemma
  - joint with Emily Riehl and Jonathan Weinberger
  - emilyriehl/yoneda
  - <u>arxiv/2309.08340</u>
- Started HoTT Book formalisation (for students):
  - <u>rzk-lang.github.io/hottbook</u>

### ITP-2023 highlight #3: Rzk and sHoTT (after)

- Rzk version 0.6.4 (latest is 0.6.7) 8 releases in 8 working days :)
- Rzk Language Server integrated with VS Code (thanks to Abdelrahman)
- 15+ active users, see contributions to <u>rzk-lang.github.io/sHoTT</u>
  - <u>rzk-lang.github.io/sHoTT/CONTRIBUTORS</u>
- 30+ participants in Rzk Zulip: <u>rzk-lang.zulipchat.com</u>
- Many people (more than I expected) interested in Rzk and its further development
- Many directions for future work:
  - (Directed) Higher-inductive types, Modalities,  $(\infty,\infty)$ -categories
  - Better IDE support