# Как решить рекурсивное уравнение

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# Sample Olympiad problem & answer

- Let $\boldsymbol{Z}$ be the set of integers. Determine all functions $f: \boldsymbol{Z} \to \boldsymbol{Z}$ such that, for all integers $a$ and $b$, $f(2a) + 2f(b) = f\big(f(a+b)\big)$.
- The problem can be solved using classical monadic recursion elimination technique known in Theoretical Computer Science since late 1960: $f(x) = 2x + const$.

# Problem via recursion elimination

- A classic example monadic recursion elimination by reduction to the tail recursion is a so-called John McCarthy function $M_{91}: \boldsymbol{N} \to \boldsymbol{N}$:

$$M_{91}(n) = if\ n > 100\ then\ (n - 10)\ else\ M_{91}\big(M_{91}(n + 11)\big).$$

- It was introduced by John McCarthy, studied by Zohar Manna, Amir Pnueli, Donald Knuth. It turns out that

$$M_{91}(n) = if\ n > 101\ then\ (n - 10)\ else\ 91.$$

# Problem via recursion elimination

- A "key" idea elimination is a move from a monadic function $M_{91}: \boldsymbol{N} \rightarrow \boldsymbol{N}$ to a binary function $M2: \boldsymbol{N} \times \boldsymbol{N} \rightarrow \boldsymbol{N}$ such that for all $n, k \in \boldsymbol{N}$

$$M2(n, k) = (M_{91})^k(n)$$

where $(M_{91})^k(n)$ is $k$-time application of the function, i.e.:

- $(M_{91})^k(n) = M_{91}(\ldots M_{91}(n) \ldots),$
- $M2(n, 0) = (M_{91})^0(n) = n$ for all $n \in \boldsymbol{N}.$

# Why to solve recursive equations?

- To contribute to IMO Grand challenge
- To test compilers for functional languages (as $M_{91}$)
- To optimize recursion implementation via recursion elimination
- To make level of program languages higher (by implicit functions)

# A fresh Olympiad problem and its programming interpretation

- Let $\boldsymbol{Q}^+$ be the set of positive rational numbers.
  - Problem: Determine all functions $f: \boldsymbol{Q}^+ \to \boldsymbol{Q}^+$ such that, for all positive rational $a$ and $b$ holds $f\big(af(b)\big) = {}^{f(a)}/_{b}$.
  - Interpretation: Implement a function $f: \boldsymbol{Q}^+ \to \boldsymbol{Q}^+$ that is not a constant such that, for all positive rational $a$ and $b$ holds $f\big(af(b)\big) = {}^{f(a)}/_{b}$.